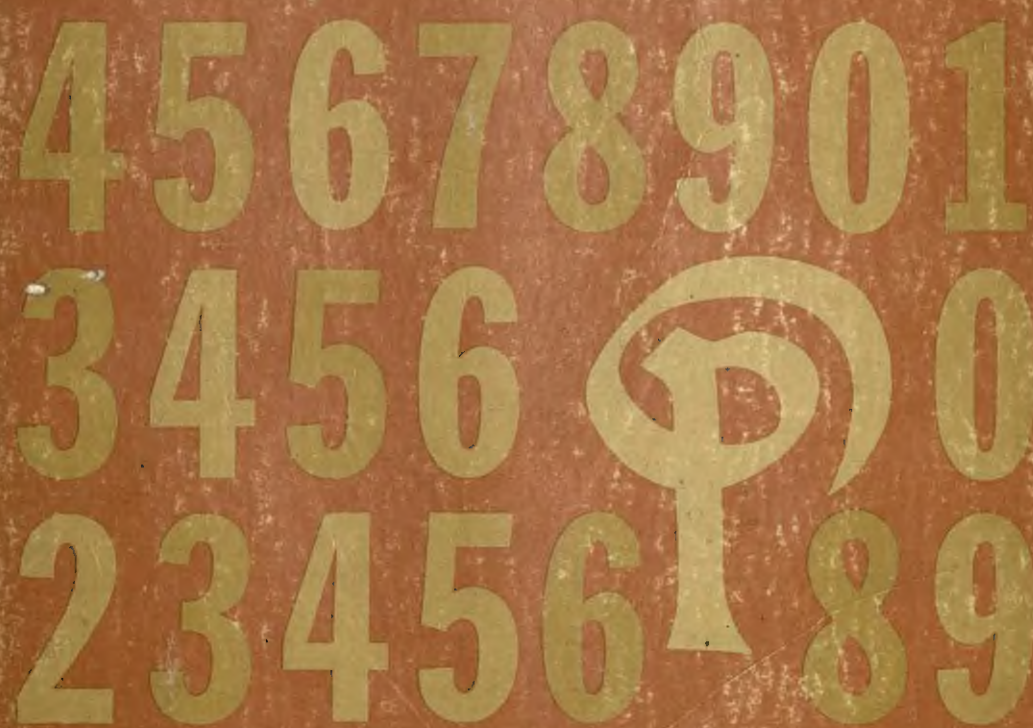


# Extending Understandings of Mathematics



*Roger Osborn*

*M. Vere DeVault*

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EXTENDING  
UNDERSTANDINGS  
OF  
MATHEMATICS

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*under the editorship of*

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The mathematical symbol used on the cover of this book is the familiar addition sign as it was written by the Renaissance mathematician and calculator Tartaglia. It is the first letter of the Italian *piu* (plus).

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056

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## Preface

*Extending Understandings of Mathematics* is a substantial revision of earlier editions published as *Toward Improved Understanding of Mathematics, Part II*, and *Extending Mathematics Understanding*. The materials contained in these chapters were originally prepared as part of an extensive in-service education project designed to provide initial study of new mathematics for elementary school teachers. Since the first publication of these books in 1960, a large proportion of elementary teachers have extended their understanding of mathematics in a variety of ways. In-service education programs, college courses, lectures, and the use of new mathematics in the classroom have all contributed to the increased mathematical knowledge of elementary teachers. Elementary teachers are the first to admit, however, that the job of improving mathematics instruction in the schools is a continuing one which can be expected to occupy the attention of elementary teachers in the years ahead.

A large proportion of prospective elementary teachers completing their undergraduate work have credit in mathematics. As a group, they have experienced a wide range in the number of course credits and of course types. In some teacher education programs as many as twelve credits of mathematics are required; in others as few as two or three credits in methods meet the requirements. The mathematics courses vary widely also in their content and in the precision and vigor of the mathematics presented. This diversity in teacher preparation results in a wide variety of understanding on the part of teachers in our schools.

The present volume is designed to provide a treatment of mathematical topics in a manner which is mathematically precise but which can be easily read and understood by elementary teachers with a wide variety of backgrounds. Throughout the volume many of the mathematical discussions are set in the context of the elementary school and thereby directly relate to the experience and instructional needs of the teacher working with children in today's classrooms.

*Extending Understandings of Mathematics* includes historical materials, measurement and geometry, statistics, and applications. As such it is designed to extend the understandings teachers have of the mathematics which they teach in the elementary school. *Understanding the Number System*, a complementary volume, gives attention to the topics of sets, number and numeration systems, the properties and operations with both whole numbers and fractions, and with equivalence relations. Since the 1961 (combined) edition, major editorial revisions have been made in each chapter; new chapters in Geometry and in Classes of Numbers have been added; and the material is published as two paperbacks rather than as one casebound volume. The latter decision resulted from the belief that teachers' understandings have improved to a point where content texts will increasingly be needed as references used in close conjunction with methods texts. The methods courses which for the past several years too frequently have been devoted largely to content are now likely to assume basic content information and proceed from there with a thorough treatment of the problems of curriculum and instruction. Even though this assumption is made, it is understood that many students and many teachers will not have the content background needed to make the methods work meaningful, and these teachers need materials which will assist their individual efforts to build the necessary background. These two volumes are designed to assist these efforts.

Roger Osborn  
M. Vere DeVault  
Claude C. Boyd  
W. Robert Houston

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EXTENDING  
UNDERSTANDINGS  
OF  
MATHEMATICS

# Chapter 1

Concepts to be developed in this section are:

1. *Systems of numeration are man-made.*
2. *Our numeral system has developed over a long period of time.*
3. *Some understanding of other systems of numeration will enhance our understanding of our own system.*
4. *Children can be led to appreciate the utility of the modern decimal system by learning about other systems.*

# Developing Historical Perspectives

## I. SYSTEMS OF NUMERATION

It should be obvious that it would be quite impossible to write a complete history of anything. The very nature of history precludes completeness. And yet, in the case of systems of numeration, the subject is such that it does not predate recorded history, for numeration is itself a recording of information. Even with the assurance that many of the facts of the history of numeration have been preserved, the great number and diversity of systems which have been used forces us to consider only a small portion of this history.

We must recognize that systems of numeration are man-made, as is all of mathematics. The symbols which man has used to record his mathematics have been symbols which he devised, and they have changed and evolved throughout a very great part of recorded history into the forms with which we are familiar today.

In studying systems of numeration, we could use a chronological approach, a geographical approach, an approach based on types of civili-

zation, on characteristics of the systems themselves, or any one of a number of other possibilities. We will pursue our study from a standpoint of the characteristics of the systems themselves and will insert comments about various civilizations (which used particular systems) and their chronology.

Man learned to count long before he learned to symbolize the counting numbers. The history preceding his learning to count is not of consequence here, nor, for the most part, is the period of time during which he learned to count but had not yet learned to write, draw, or scratch symbols representing numbers. Only a few aspects of this presymbolic period are of real consequence to us here. One of these aspects is man's use of his fingers, or pebbles, or sticks to represent numbers before he began to use written symbols. There seems to be little doubt that the use of one finger or one stick to symbolize the number one substantially influenced the use of a stroke (horizontal or vertical) as the numeral representing the number one. Similarly, two and three strokes ( $||$  or  $=$  and  $|||$  or  $\equiv$ ) seem surely to be tied to the use of fingers or sticks. You will note how our numerals 2 and 3 could have evolved from the  $=$  and  $\equiv$ . While these explanations are based on conjecture, they do seem to be the only reasonable interpretation of the observed facts of many systems. The backgrounds for the use of other symbols are much more obscure. We shall direct our attention, then, to the observed facts about systems of numeration and put aside most of the conjectures about these systems.<sup>1</sup>

## TYPES OF NUMERATION SYSTEMS

Systems of numeration may be grouped roughly into four types. These types may be described as

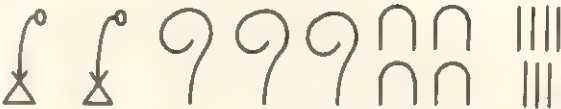
- (1) additive or simple grouping,
- (2) multiplicative,
- (3) ciphered,
- (4) positional or place value.

The simplest of the additive or simple grouping systems was historically much like the system we use today to keep score in certain

---

<sup>1</sup> Further reading in this area may be done in any one of a number of excellent books, among them: Howard Eves, *An Introduction to the History of Mathematics*, Second Edition (New York: Holt, Rinehart & Winston, Inc. 1964); James R. Newman, *The World of Mathematics* (New York: Simon & Schuster, Inc., 1956), I; David Eugene Smith, *History of Mathematics* (New York: Dover Publications, Inc., 1958), I.

games. We have all seen a numeral indicated as  $\text{||||} \text{||||} \text{||}$ . Early additive systems show traces of this type of numeration, and even complicated systems fall into this category. The early Egyptian hieroglyphics, which go back to before 3000 B.C., provide an example of simple grouping and addition. A vertical mark (or staff) represented one, a heel bone (inverted U) represented ten, etc. A few of these symbols are shown in Figure 1-1. In such a system the values of all the symbols in a numeral are added. Thus, our numeral 2,347 would have appeared as:





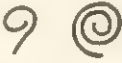

EXAMPLES OF EGYPTIAN SYMBOLS		
1		staff
10		heel bone
100		scroll (or coiled rope)
1000		lotus flower

FIGURE 1-1

Other such systems, using different symbols, were recorded in the cuneiform inscriptions of the Sumerians (*circa* 3000 B.C.) and the early Babylonians (*circa* 2000-200 B.C.). The Herodianic Greek system (*circa* 400 B.C.) was also additive. Of course, the Roman system, both in its early form and in its modern equivalent, is an additive or simple grouping system. In its modern form, as in certain other early systems (for example, the Babylonian), a subtractive principle was also allowed. In the Roman system this is indicated by a symbol of lesser value preceding one of greater value (see Figures 1-2 and 1-3).

The second general category into which some systems may be classified is the multiplicative system. In such a system, two sets of symbols must be used. One set consists of the basic digits, for example, 1, 2, 3, . . . , 9, and the other set consists of symbols representing powers of the base, say, 10, 100, . . . In this system the symbols from each group are written alternately, the first of each pair of sym-

EXAMPLES OF ROMAN SYMBOLS	
1	I
5	V
10	X
50	L
100	C
500	D
1000	M

FIGURE 1-2

bols telling how many of the units represented by the second symbol are to be taken. All results are then added. Figure 1-4 illustrates the principle, although the symbols bear no relation to any historical system. The traditional Chinese-Japanese numeral system was a multiplicative system. It dates back to antiquity. The symbols used were more complicated than those used in the example; the base was 10, and the numeral was written vertically from top to bottom.<sup>2</sup>

A ciphered numeral system uses even more symbols. If  $b$  is the base, it requires symbols for  $1, 2, 3, \dots, b-1; b, 2b, 3b, \dots, (b-1)b; b^2, 2b^2, \dots$ . Representations of numbers are compact after the

EXAMPLES OF ROMAN NUMERALS		
Our Numeral	Modern Roman	Ancient Roman
1944	MCMXLIV	MDCCCCXXXIIII
1963	MCMLXIII	MDCCCCLXIII

FIGURE 1-3

<sup>2</sup> A table of these symbols is shown in Eves, *An Introduction to the History of Mathematics*, p. 13.

lengthy table of numerals is memorized. The value of the number represented is the sum of the values of the symbols used. The Ionic Greek system (*circa* 400 B.C.) was a ciphered numeral system. This system used portions of the Greek alphabet to represent each set of numerals. The accompanying illustration and examples in Figure 1-5 are given in the English alphabet (with "&" included to make a twenty-seventh letter). Still more symbols would be required to represent numbers greater than 999.

SAMPLE MULTIPLICATIVE SYSTEM	
1 —	5 F
2 ≡	25 T
3 X	
4 □	

EXAMPLE	
Our Numeral	Sample Multiplicative System
$97 = 3 \times 25 + 4 \times 5 + 2$	X T □ F =

FIGURE 1-4

The positional numeral system, which includes our own system of numeration, is the one with which we are most familiar. The Babylonians began evolving a place value (positional) system somewhat before 2000 B.C. In this system the base was 60. For numbers greater than 60 the positional principle was applied, but for numbers less than 60 simple grouping was used. A subtractive principle was also used, although the examples given in Figure 1-6 do not employ it. It should be quite apparent that such a system needs a zero to be a place holder—an attempt to write the number 7203 in Babylonian form fails if no place holder is used. The early Babylonians had no such numeral, and

A CIPHERED NUMERAL SYSTEM					
1	a	10	j	100	s
2	b	20	k	200	t
3	c	30	l	300	u
4	d	40	m	400	v
5	e	50	n	500	w
6	f	60	o	600	x
7	g	70	p	700	y
8	h	80	q	800	z
9	i	90	r	900	&

EXAMPLES	
Our Numeral	Ciphered Numeral
15	je
194	srd
347	umg

FIGURE 1-5

their system of numeration did not flourish until the zero was introduced.

The Maya Indians also had a system of numeration (of remote origin) involving the positional principle, but, probably because their year contained 360 days, their building blocks were 20,  $18(20)$ ,  $18(20^2)$ , etc. They also used a symbol for zero. Thus, in the Mayan system the numeral representing 26,656 would have been represented as  $3 \times 18(20^2) + 14 \times 18(20) + 0 \times (20) + 16$ . The accompanying list of symbols (Figure 1-7) illustrates that the simple grouping technique

was used for numbers less than 20. For numbers greater than 20, the positional principle was used, the numeral being written vertically with the top symbol being the one of greatest place value.

It hardly seems necessary to say much about our own Hindu-Arabic system as far as structure is concerned. The ideas and digits used had their origin in ancient India and Babylon. The system was probably brought to Europe by traders and Arabic invaders during the crusades. The present forms of the digits have evolved primarily in Europe. Figure 1-8 shows some of the steps in the evolution of our digits. These steps are shown to illustrate that our system of numera-

BABYLONIAN SYMBOLS		SIMPLE GROUPINGS	
1		5	 
10	<	13	<
		32	<<<<

BABYLONIAN POSITIONAL NUMERALS	
Our Numeral	Babylonian
18,812	<   <<<<
$= 5(60)^2 + 13(60) + 32$	

FIGURE 1-6

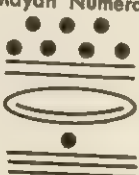
tion is still evolving. Even today manufacturers of type make many different type faces. You need only to look at typewritten material done on two different typewriters to know that not all digits look alike. The current trend is toward a typeface which is very easy to read. It may be that a century from now the common appearance of our numerals will have changed considerably.

The naming of numbers is an interesting feature of our system of numeration. We all know that number names vary from language to

language, but it is not so well known that even in our language the same name is not used for the same number by all English-speaking people. A notable example of this is the name "billion." In the United States the name "billion" refers to one-thousand million, but in Great Britain it refers to one-million million.

Our English language has another peculiarity which is largely overlooked. Our system of numeration is of the positional or place value type, but our number names are of the multiplicative type discussed previously. The multiplicative type numeral has two sets of symbols—one set tells how many of some type unit are to be taken, and the other set tells what the units (powers of the base) are. In our number names, the two sets of symbols are: (1) number names, for numbers less than one thousand, which are read and spoken according to the positional value of the digits, and (2) the unit names: thousand, million, billion, etc. (in which the base is 1000, and the names describe the successive powers of 1000). Several examples, shown in Figures 1-9 and 1-10, should serve to illustrate this feature of our number names.

As one last comment about this peculiarity of our system of number names, let the first set of symbols be the numerals 0, 1, 2, 3, . . . , 999, and let the second set of symbols be *T*, *M*, *B* for thousand, million, and billion, respectively. Then the numerals of Figure 1-10 would be named as indicated in Figure 1-11.

EXAMPLE	
Our Numeral	Mayan Numeral
26,656	

MAYAN NUMERALS	
1	•
2	• •
3	• • •
4	• • • •
5	_____
6	_____ •
7	_____ • •
8	_____ • • •
9	_____ • • • •
10	=====
11	===== •
12	===== • •
13	===== • • •
14	===== • • • •
15	=====
16	===== •
17	===== • •
18	===== • • •
19	===== • • • •
0	○

FIGURE 1-7

YEAR	1	2	3	4	5	6	7	8	9	0
976 A. D.	I	7	z	4	U	L	7	8	9	
1294	1	2	3	4	U	6	7	8	9	0
1442	1	2	3	4	4	6	^	8	9	0

FIGURE 1-8

## CREATION OF SYSTEMS OF NUMERATION

In the classroom today, the child may achieve a better understanding of our system of numeration if he is given an opportunity to create his own system. Let us look at a system which a fifth-grade child created for use as numerals on a clock face. He wanted to create a base twelve system. The accompanying sketch and table (Figure 1-12) give the essence of his system.

Let us take a critical look at his system to consider its features and its possibilities for use as a system of numeration. First, we can see that the symbols shown in both the clock face and Figure 1-12 are the

NUMBERS LESS THAN 1000—NUMBER NAME DETERMINED BY PLACE VALUE OF DIGITS		
NUMERAL	NAME	MEANING
8	eight	eight ones
72	seventy-two	seven tens and two ones
346	three hundred forty-six	three hundreds, four tens, and six ones
507	five hundred seven	five hundreds, no tens, and seven ones
823	eight hundred twenty-three	eight hundreds, two tens, and three ones

FIGURE 1-9

NUMBERS GREATER THAN 1000— NUMBER NAME READ ACCORDING TO MULTIPLICATIVE PRINCIPLE	
NUMERAL	NAME
3,456	{three} thousand {four hundred fifty-six}
432,198	{four hundred thirty-two} thousand {one hundred ninety-eight}
262,393,456	{two hundred sixty-two} million {three hundred ninety-three} thousand {four hundred fifty-six}
42,262,393,456	{forty-two} billion {two hundred sixty-two} million {three hundred ninety-three} thousand {four hundred fifty-six}

FIGURE 1-10

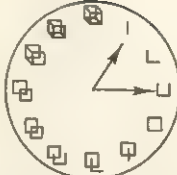
same. In reading these numerals there may be some perceptual difficulties. This, however, is not a defect which would make the system unusable.

Second, we note that no zero has been included in the system. The lack of a zero is probably a fatal defect. All systems (except the simple grouping system) which have had no zero have passed out of general usage, and it is certain that the lack of the zero has had a large part in their passing. Our fifth grader can correct this defect, of course, by introducing a zero into his system. We will assume hereafter that a zero has been introduced.

A third aspect of our consideration of this system should be our decision of how to extend the system. Do we want a positional system, a multiplicative one, a system of ciphered numerals, or merely an additive system? If we want any of these systems except the additive we must have a base. The boy who devised this system wanted the base to be 12. Let us follow this decision, and consider its implications. Figure 1-13 shows the changes we must make.

NUMBER NAMES READ ACCORDING TO MULTIPLICATIVE PRINCIPLE	
Numeral	Name of Number Read as if Written as
3,456	3T 456
432,198	432T 198
262,393,456	262M 393T 456
42,262,393,456	42B 262M 393T 456

FIGURE 1-11



Conventional Numeral	Boy's Numeral
1	
2	└
3	└
4	□
5	□
6	□
7	□
8	□
9	□
10	□
11	□
12	□

FIGURE 1-12

These changes are required for the following reasons: (1) in a positional system, there is no separate symbol for the base; (2) in a multiplicative system, symbols are needed for each separate power of the base; (3) in a ciphered system, symbols are needed for each multiple of each power of the base.

Let us adopt the additional symbols needed (illustrated in Figures 1-14 and 1-15) and use them with those of Figure 1-12.

Any of these three systems is now adequate for representing any number smaller than ten thousand. See Figure 1-16 for the representations of several numbers in several systems. The child can readily see that there are disadvantages to the third and fourth systems shown in Figure 1-16. The fourth is compact, but it requires too extensive a


POSITIONAL	Abandon the symbol  . Include the symbol $\bigcirc$ .
MULTIPLICATIVE	We need symbols for 144, 1728, . . .
CIPHERED	We need symbols for 24, 36, 48, . . . , 132; 144, 288, . . . , 1584; 1728, . . .

FIGURE 1-13

MULTIPLICATIVE
$144 = y$
$1728 = z$

FIGURE 1-14

table of symbols to be memorized. There seems to be nothing at all to recommend the multiplicative system. The base 12 positional system will seem to have no particular advantage over the ordinary base 10 system, and the child—through this observation—may gain an appreciation for our own familiar system.

There is one remaining facet of numeration in any system which should be considered. This is the use of the system for computation. In order to compute, certain addition and multiplication facts must be used. These facts can best be organized by the construction of addition and multiplication tables.

## II. THE EXPLODING UNIVERSE OF MATHEMATICS

Concepts to be developed in this section are:

1. *Mathematics is man-made.*
2. *Mathematics is dynamic and growing.*
3. *There are two kinds of mathematics, two kinds of mathematical activity.*

Mathematics in the twentieth century is a subject which has had a rate of growth so great that a title embodying the words "changing"


CIPHERED		
12 = 	144 = m	1728 = x
24 = b	288 = n	3456 = y
36 = c	432 = o	5184 = z
48 = d	576 = p	.
60 = e	720 = q	.
72 = f	864 = r	.
84 = g	1008 = s	
96 = h	1152 = t	
108 = i	1296 = u	
120 = j	1440 = v	
132 = k	1584 = w	

FIGURE 1-15

or "growing" or even "expanding" would be inappropriate. It can almost literally be said that the whole universe of mathematics is *exploding*, so rapid is its expansion in scope and depth.

It has been said that the nineteenth century contributed some five times as much to mathematics as the total contribution of all the nearly sixty centuries of recorded history of mathematics which preceded it. It remains to be seen what the total contribution of the twentieth century will be, but it is safe to say that new contributions to the field of mathematics since 1920 would very nearly balance all of mathematics preceding that date. The curve of development is rising very sharply, and there is no indication that it will falter in the foreseeable future.

Oftentimes the question is asked, "Is there really anything new in mathematics since the time of the Greeks?" The answer is, emphatically, "Yes!" Only a few fields (of the multitude available) need be named. The Greeks had no abstract algebra, no functions of a complex variable,

























NUMERALS IN FOUR SYSTEMS			
BASE TEN (ORDINARY)	BASE TWELVE POSITIONAL SYSTEM	BASE TWELVE MULTIPLICATIVE	BASE TWELVE CIPHERED
8			
12	0		
29	L 	L  	
505	 	  	
3,947	L    	L     	

FIGURE 1-16

no probability theory. Many people may think that a modern mathematician merely makes new patterns out of dead bones in a graveyard. To show that such is not the case is one of the concerns of this chapter. The building blocks of today's mathematician may be very ancient, but they are still extremely useful. The structure that is built with them is dynamic and growing.

Many people have made contributions, some great and some small, to this growth of mathematics. To write a complete history of these contributions is patently impossible. In this section we will look at an outline of these contributions and the nature of some of them. In another section, we will take a closer look at what kind of men contributed to mathematics; we will look at the men themselves and their individual contributions.

## TWO KINDS OF MATHEMATICS

In his little book, *The Nature of Mathematics*,<sup>3</sup> Philip Jourdain (1879-1919) has proposed that there are two kinds of mathematics. He distinguishes between "mathematics" which consists of the methods used to discover certain truths and "Mathematics" which consists of

<sup>3</sup> Reproduced in its entirety in Newman, *The World of Mathematics*, Vol. I, pp. 4-72.

the truths themselves. We can see that the methods used to discover certain truths may involve relations with and reference to physical reality, while the truths themselves are abstract, not necessarily bearing relationship to physical reality. Herein is seen some of the division between the so-called pure and applied branches of mathematics. While some of us may not agree with Jourdain's division of mathematics, we will undoubtedly find that it stimulates us to consider a new point of view.

There are also two kinds of mathematical activity (aside from teaching, learning, and using mathematics) in which mathematicians engage. These two types of activity are not parallel in all respects to the two kinds of mathematics just mentioned, but they do have some parallel features. These two kinds of activity might be characterized as "pioneering" and "developmental." By "pioneering" we mean those activities which involve exploring new territory, pushing out the frontiers, and creating systems not previously known. "Developmental" activities are those in which the rough edges are smoothed, in which the holes and gaps are filled, in which new applications of both new and old systems to the physical world are found and explored, and in which new systems are explained and taught to others. These two names were selected to correspond to the pioneer and the developer in other phases of our society.

In mathematics, the same mathematician may engage in both types of activity, or he may be a master of one and be poorly equipped (psychologically or otherwise) to do the other. There are many examples of men who have been both great researchers and great teachers, but it is unfortunate that the requirements of one activity may interfere seriously with the other. A classic example of the pioneering type, who detested developmental activity, was the Frenchman, Evariste Galois (1811-1832). In spite of having wasted a good part of his short life, Galois wrote down—on the night before the duel which cost him his life at age twenty—the foundations for much of the modern topic of group theory. He made such monumental contributions to the beginnings of modern algebra that his mathematical heirs have been kept busy for more than a century in the development of the subjects he pioneered in such a short time.

## THE MAN-MADENESS OF MATHEMATICS

For centuries the controversy has waxed and waned—is mathematics invented or discovered? Is its structure purely a creation of the mind of man or is it superhuman, awaiting discovery by man? Whichever view

## ARITHMETIC

Beginnings before 4000 B.C. (Egyptian calendar).

Various systems for performing commercial arithmetic,  
4000 B.C. to 1500 A.D. (Egyptian, Babylonian,  
Greek, Hindu, Arabic, Mayan, etc.)

Zero in common use since about 1000 A.D.

Fully developed positional system of numeration in  
use since about 1500 A.D.

Machines to do computation in use since about 1900.

Electronic computers (high speed) in use since about 1950.

FIGURE 1-17

we take, it is impossible to rule out the role of man in the pioneering and developmental activities associated with mathematics. If we say that some man discovered a new principle, we are not trying to give an answer to the age-old questions above, but we are trying to describe the pioneering activity in which he was engaged. We might just as well have described his initial recognition of this principle by saying that he was the first to create this statement of principle.

Only man, of all the animals on earth, has developed mathematical systems, and only through the actions of the minds and hands of man—individually, collectively, and cumulatively—has this mathematical structure grown. The historical outlines in Figure 1-17 are developed along three separate lines—arithmetic, algebra, and geometry. Around the middle of the seventeenth century the paths of the three began to merge into one path, which thereafter could be called mathematical analysis. Throughout the course of history there have been offshoots from these mainstreams, but almost invariably these offshoots have later been included in some larger interpretation or theory and have thereby rejoined the mainstream. This very brief outline of the history of arithmetic goes back to at least 4000 B.C., at which time a well-developed Egyptian calendar was already in existence, implying that mathematical knowledge was even then on a fairly advanced plane. Similarly, the outline of geometry goes back to the Egypt of about 3000 B.C., a date when some knowledge of geometry was necessary for the re-establishment of boundary lines after the floods of the Nile.

**ALGEBRA**

First developments as a subject separate from arithmetic around 600 A.D.

Development of adequate symbolism:

+ and - about 1490,

= about 1540,

modern exponent notation about 1630.

Beginnings of treatment of algebra as a logical system with adequate notation about 1800.

Generalizations (group theory, abstract algebra, etc.)  
principally in the 20th century.

**GEOMETRY**

Beginnings in Egypt before 3000 B.C.

Euclid's work of systematizing geometry as it was known in his time,  
about 300 B.C.

Commentaries written and problems worked—no pioneering—  
from 300 B.C. to 1800 A.D.

Beginnings of non-Euclidean geometries (Lobachevsky, Bolyai, Riemann)  
about 1825 A.D.

Placing geometry on rigorous foundations, a development of the  
20th century.

FIGURE 1-17—*Continued*

Hidden by the brevity of these notes are some of the major achievements which led to the astounding growth of modern mathematics. Aside from the main currents of mathematical thought which can be traced far back into the past, some of the keys to expansion were: (1) the development of notation adequate to express problems and their solutions, (2) the development of non-Euclidean geometries which gave man a push enabling him to break with past conceptions of geometry and the physical world, (3) the development of analytic geometry (just over 300 years ago) and the calculus (less than 300 years ago),

and (4) the search for the development of rigorous bases for many of the main subdivisions of mathematics.

Of these four keys, the one which probably affects the greatest number of us in the elementary classroom today is the first. The words of Philip Jourdain emphasize this point:

It is important to realize that the long and strenuous work of the most gifted minds was necessary to provide us with simple and expressive notation which, in nearly all parts of mathematics, enables even the less gifted of us to reproduce theorems which needed the greatest genius to discover. Each improvement in notation seems, to the uninitiated, but a small thing; and yet, in a calculation, the pen sometimes seems to be more intelligent than the user. Our notation is an instance of that great spirit of economy which spares waste of labour on what is already systematised, so that all our strength can be concentrated either upon what is known but unsystematised, or upon what is unknown.<sup>4</sup>

The body of mathematical knowledge—the universe of mathematics which man has accumulated over many centuries through the thought and effort of many—is so great today that no one man can know it all. In fact, it has become so complex that only years of superior training can equip a man to engage successfully, and in a worthwhile manner, in developmental or pioneering activities in even a small fraction of this universe. For this reason, it is essential that each of us who will convey this mathematical knowledge to the next generation prepare himself for exercising this responsibility to the best of his ability.

It cannot be too strongly emphasized that all mathematical achievements—whether pioneering or developmental—are those of man, using the mind and hands with which he is endowed. Hence, we conclude that the universe of mathematics is man-made.

### III. CREATORS OF MATHEMATICS

The concept to be developed in this section is:

*Men of many chronological ages, historical ages, and stations of life have created mathematics.*

If we think of two opposing categories into which members of the human race may be placed, we will probably think of two categories

<sup>4</sup> In Newman, *The World of Mathematics*, Vol. 1, p. 13. Reprinted by permission.

into which some two great mathematicians may be classified. Mathematicians, like other people, come in many sizes, with many temperaments, from many backgrounds. In this section we will look at some of the world's greatest mathematicians and the fields in which they worked. Few men of the twentieth century will be mentioned because, for the most part, history has not yet evaluated their work.

To the non-mathematician any discussion of the work of a very great mathematician may be nearly meaningless. The universe of mathematics has become so complex that it may be true that even one mathematician may understand little, if any, of the work of another. In this section we will try to be descriptive rather than technical in our brief glimpses of the work of some of the world's greatest mathematical creators.

## AGES OF HISTORY AND AGES OF MEN

Men of many historical ages and of many chronological ages have expanded the vast domain of our mathematical knowledge. We have already noted that there was a well-developed Egyptian calendar before 4000 B.C. This certainly implies an advanced system of arithmetic. Certainly, it is known that the structuring of mathematics began in the dark reaches of prehistory. Because the recorded history of mathematics does reach so far back we cannot name particular men nor their individual contributions. We only know that men of this prehistoric age did further mathematical knowledge. We do have records of the names and ideas of men who contributed long before the Christian era. All of us have heard of Euclid, the great Greek geometer who brought together and systematized the geometric knowledge of his day. He lived some 300 years before Christ. Probably most of us have not heard of the Greek merchant and statesman, Thales of Miletus (born *circa* 640 B.C.), who preceded Euclid by some 300 years. He contributed to mathematical astronomy. Nor have many of us heard of an Egyptian of an even earlier date, whose name is known to us as Ahmose or Ahmes. His claim to mathematical fame is that he is the scribe who copied in about 1650 B.C. an older treatise on mathematics. His papyrus manuscript has come down to us and is known to many as the Rhind Papyrus.

Among the mathematicians of antiquity, one stands out above all the others. This man was Archimedes (born, Syracuse, Sicily, 287 B.C.; died, Syracuse, 212 B.C.). He ranks among the three greatest mathematicians of all time (Newton and Gauss being the others, and they lived some 1900 and 2000 years later, respectively). Archimedes' greatness in

mathematics extended to many fields, the principal ones being arithmetic, algebra, geometry, and the application of mathematics to mechanics. He anticipated many of Newton's discoveries in mechanics, worked with infinite series, used (in a primitive form) a law of exponents, did some work in analytic geometry (1900 years before Descartes), and most amazing of all, used some methods which are very much like those of the calculus which was not developed until nearly 1900 years later.

While it is impossible to mention all who contributed to mathematics during the course of history, it might be of interest to note that not all mathematical activity was centered in Europe and Egypt. In fact, during the Dark Ages (500-1000 A.D.) in Europe, while what interest there was in learning was barely kept alive in the monasteries, mathematical activity of a creative type was being carried on in many other places. In 662 Severus Sebokht, a Christian bishop in Mesopotamia, wrote a treatise describing the Hindu numeral system; this is one of the earliest such descriptions to have come down to us. An Arab mathematician, Mohammed al-Khowrizimi, wrote the first work bearing the title "algebra" around the year 800.

From the close of the Dark Ages to the present, no century has been without its contributors to the ever expanding universe of mathematics, although, as we shall see, men of some centuries have had a greater hand in the molding of mathematics than have men of other centuries. The names and some of the contributions of these men will be found in subsequent paragraphs.

Contributions to mathematics have been made not only by men of all historical ages, but also by men of all chronological ages. All the contributions of Evariste Galois to the fields of algebra and group theory were made before he was twenty-one. Carl Frederick Gauss (1777-1855) made his first contributions—the proof of the general binomial theorem and the discovery and proof of the law of quadratic reciprocity—before he was nineteen. The account of his life in E. T. Bell's *Men of Mathematics* (New York: Simon and Schuster, Inc., 1937) makes fascinating reading, for in it we are able to see the unfolding and flowering of one of the world's greatest geniuses.

One of our great mathematicians has remarked that a mathematician may write mathematics itself before he is forty, but that after forty he only writes about mathematics. This is not necessarily the case as can be seen in the career of J. J. Sylvester (1814-1897). He lived a long and fruitful life, and he made distinctly valuable contributions to the subject of compound partitions (an area of number theory) when he was in his eighty-second year.

Another mathematician, Jacob Steiner, 1796-1893, made notable contributions to geometry when he was seventy-one years of age.

## PROFESSIONS OF MATHEMATICIANS

Although it can be said that the profession of a mathematician is mathematics, it also is true that mathematics has been created by men from many other professions. Only a few will be named, but there are many others. The examples which follow are from many periods of history (although few are contemporary because most mathematics today is being created by professional mathematicians); they are chosen somewhat randomly, not because the man was the greatest in his field.

### PROFESSION

### MATHEMATICAL CREATOR

MEDICINE

Nicholas Chuquet (died *circa* 1500) was French. He contributed to arithmetic (rational and irrational numbers) and to algebra (theory of equations).

LAW

Francois Viete (1540-1603) was also French. He worked primarily in algebra and applied algebra to geometry in such a way as to lay foundations for modern trigonometry.

LAW

Pierre de Fermat (1608-1665) was one of the greatest number theorists who ever lived. His correspondence with his fellow French mathematicians included some elements of analytic geometry a number of years before Descartes published his first work on the subject.

LAW

Arthur Cayley (1821-1895) was an English mathematician of note who took up and practiced law in order to earn a living. He wrote one book, *Treatise on Elliptic Functions*, and more than a thousand papers covering a wide range of mathematical topics, chief of which was that of the theory of invariants.

LAW

James R. Newman (1907- ) was born and has lived in the United States. One of his great contributions has been that of making mathematics understandable (to some extent) to the

layman. His work includes *The World of Mathematics* (4 vols., New York: Simon and Schuster, Inc., 1956), of which he was editor, and *Mathematics and the Imagination* (with Edward Kasner; New York: Simon and Schuster, Inc., 1940).

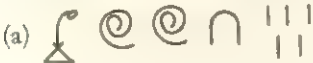
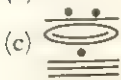
- ARCHITECTURE** Sir Christopher Wren (1632-1723) would have been known as a mathematician if he were not better known as the architect who designed St. Paul's Cathedral after the great fire of London. His mathematical contributions were in the fields of geometry, astronomy, and mechanics.
- ENGINEERING** Gerard Desargues (1593-1662) was French. His contributions to pure geometry were the most outstanding of the seventeenth century.
- THEOLOGY** Baeda, or Bede the Venerable (*circa* 673-735), was a great Church scholar, an English monk, who contributed to number theory and to the study of the ecclesiastical calendar. His was the best work up to his day on digital notation.
- THEOLOGY** William Oughtred (1574-1660) was an English clergyman whose main interest was mathematics. He contributed to arithmetic and its notation and to trigonometry. His invention of the slide rule is probably his best-known work.
- ASTRONOMY** The names of Galileo Galilei (1564-1642) and Johann Kepler (1571-1630) are well known. Their contributions to mathematical astronomy cannot be overrated. They also both contributed to geometry.
- POETRY** Omar Khayyam (*circa* 1044-1124) is best known in the Western world as a poet. He also wrote notable Persian works on Euclid, astronomy, and algebra.
- LIBRARY SCIENCE** Eratosthenes (*circa* 274-194 B.C.) was librarian at the University of Alexandria. He contributed to arithmetic (number theory) and earth measurement.
- ARMY** Simon Stevin (*circa* 1548-1620) of Holland contributed to arithmetic, giving the first systematic theory of decimal fractions.

## ARMY

Janos Bolyai (1802-1860) was a young army officer at the time he constructed and developed (independently of Lobachevsky) a non-Euclidean geometry, helping thereby to give impetus to the movement toward the modern axiomatic treatment of mathematics.

These are but a few of the professions which have supplied representatives who have made contributions to mathematics. Since most of the great contributors to mathematics have been teachers, the profession of teaching was deliberately omitted to keep the list from extending indefinitely.

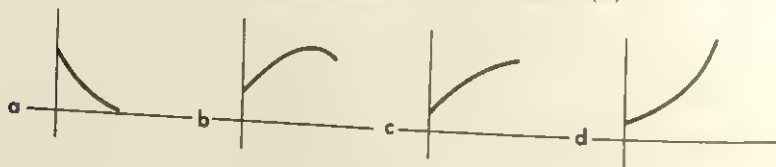
## EXERCISES

1. Represent each of the four numbers below in
  - (1) the Egyptian additive system;
  - (2) the Greek ciphered system, as represented by our alphabet, and using  $a'$  for  $1000a$ ,  $b'$  for  $1000b$ , etc.;
  - (3) the Mayan positional system; and
  - (4) the Roman additive system (with the modern subtractive principle allowed).
  - (a) twenty-seven
  - (b) three hundred forty-two
  - (c) four thousand, six hundred seventy-two
  - (d) fifty-seven thousand, nine hundred fifty-one
2. Express in the following numeration systems:
  - (1) Egyptian (2) Mayan (3) Roman
  - (a) two thousand, three hundred fifty-seven
  - (b) four hundred ninety-six
  - (c) one thousand, one hundred one
  - (d) nine hundred ninety-nine
  - (e) three hundred fifty-nine
3. Which of the four types of numeration systems listed in exercise 1 require the use of a zero in describing number?
4. "Decipher" the following numerals:
  - (a) 
  - (b) & m c
  - (c) 
  - (d) MDCXLVI

5. Create an additive system of numeration. Describe how addition, subtraction, and multiplication are performed in your system. Include tables of all facts needed.
6. Create a multiplicative system of numeration. Describe how to add and subtract in your system. Include a table of all addition facts needed.
7. Find a book describing the arithmetic of computers. Report on the system described.
8. Read and report on Philip Jourdain's book *The Nature of Mathematics*.
9. Expand the outline of the development of arithmetic, algebra, and geometry as shown on pages 16 and 17.
10. Make an outline of the development of mathematical notation.

Select the appropriate response to each of the following statements and explain the reason for your choice.

11. Which of the following is basic to all well known numeral systems?
  - (a) additive principle
  - (b) multiplicative principle
  - (c) positional notation
  - (d) grouping of elements
12. Which of the following is not involved in our own base ten numeral system?
  - (a) additive principle
  - (b) subtractive principle
  - (c) positional notation
  - (d) grouping of elements
13. If we plot "man's development of mathematical ideas" on the vertical scale and the time from 1500 A.D. to 1900 A.D. on the horizontal scale, which of the following graphs best indicates the growth of mathematical concepts during this period? (a) (b) (c) (d)



14. In man's development of a system of numeration, which of the following came first?
  - (a) place value
  - (b) numeral

- (c) zero
  - (d) cipherization
15. Which of the following was the initial motive in man's invention of numeral?
- (a) development of a system of recording
  - (b) development of a system of computing
  - (c) development of a number system
  - (d) development of a system of numeration
16. When historians look back on the entire twentieth century, in which year might we expect them to find that the most mathematics has been developed?
- (a) 1943
  - (b) 1953
  - (c) 1963
  - (d) 1973

## Extended Activities

1. Read and report on two of the men mentioned in this chapter who have contributed to mathematics.
2. Describe an abacus for representing Roman numerals and how addition may be accomplished using this abacus to represent Roman numerals.
3. Describe how this Roman abacus may be used for multiplication (equal additions).
4. G. W. Leibnitz was a contemporary of Newton's. Report on his contributions to mathematics.
5. Read and report on the Attic or Herodianic Greek system of numerals. (See Swain's *Understanding Arithmetic* or Ore's *Number Theory and Its History*.)
6. Read and report on finger reckoning. (See Ore's *Number Theory and Its History*.)
7. Find approximately how many years elapsed between the invention of the printing press and the general acceptance and use of Hindu-Arabic numerals in Europe. Give an explanation for the relation between these two events.
8. Read and report on "Calculating Prodigies" in *The World of Mathematics* by James R. Newman.
9. Read and report on the "Rhind Papyrus" and its contribution to our knowledge of ancient mathematics.

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## Chapter 2

Concepts to be developed in this chapter are:

1. *The defined unit of measure is exact; any reproduction of such a unit is an approximation.*
2. *Measurements are applications of defined units of measure and are approximate.*
3. *Both direct and indirect measurements are made.*
4. *Our units of measure were chosen by man.*
5. *Units of measure are continually evolving.*
6. *The evolution of units of measure has been in the direction of the development of standardized units.*

## Evolving Concepts of Measurement

We associate number with the qualities of material objects or phenomena in two ways. We count the occurrences of *discrete or individual phenomena*, and for *continuous phenomena*, we count the number of times some arbitrarily chosen unit is contained within the continuous whole. Counting is an integral feature of both of these ways of associating number with qualities of material phenomena. It is the second way that is usually called measurement. In this chapter our attention is directed to the process of measuring in this sense—that of finding the number of times an arbitrary unit is contained within a continuous whole.

Essential measurement concepts to be developed by elementary school children include understandings of: the meaning of measurement, measurable qualities, the approximate nature of measurements, equivalent expressions of the same measure, and the basic meanings and related skills of linear, area, volume, weight, time, and temperature measurements.

### MEANING OF MEASUREMENT

Measurement is a process of comparison which involves a known unit of measure, and a measurable quality of an object or phenomenon.

Through this process we determine the number of units contained in the measurable quality.

Measurements have degrees of directness. The most *direct* type of measurement involves only the matching of the unit of measure and the thing to be measured—for example, the cave man matching animal skins to his body dimensions. Such measurement is still in daily use. For example, the homemaker and the tailor may “custom fit” clothing directly on the person for whom the garment is being made. Likewise, we use the most direct type of measurement in putting a nail hole when we use the size of the space to be filled as the unit of measure for the putty needed. Another direct measurement involves both matching and substitution. When we apply a yardstick first to a floor to be covered, then to the floor covering material, we are making a direct measurement. This substitution idea is based on the mathematical rule that two numbers are equal to each other if each is equal to a third.

In making *indirect* measurements we use numbers obtained from the measurement of some phenomenon other than that about which the knowledge is actually desired. Time cannot be directly measured, nor can temperature. Swings of a pendulum, however, can be measured, as can the expansion and contraction of mercury in response to temperature changes. Odometers and spring scales make possible a more efficient and convenient measurement of long distances and weights, respectively, than would a tape or a balance which would permit direct measurement. On the spring scale we measure indirectly by noting the deflection of an indicator; on the odometer we measure indirectly by measuring the rotations of an indicator and relating these rotations to distance traveled.

Other indirect measurements are *derived* by the multiplication and division of quantities obtained by direct measurements of other qualities. Area of a rectangle, for example, is commonly determined by the formula  $A = LW$ , although the basic concept of area (and of other derived measures) should be developed before the use of the formula. Linear measures are also used in deriving volume:  $V = LWH$ . Measurement of speed is derived by dividing distance, a linear measure, by a time measure.

## PSYCHOLOGICAL DEVELOPMENT

Measurement concepts, like number concepts, begin developing early in life. Many of the child's first standards of comparison of size, position, and amount are egocentric. His first units of linear measure

are his own body dimensions; for example: the drawer is too high for him to reach. Similarly, his strength is his first unit for measuring weight: the big chair is too heavy for him to move, but he can carry his own little chair. He finds that he has more room (area) in which to ride his tricycle on the sidewalk than in the house, that there is not enough room (volume) for both himself and his dog in the doghouse. If he prefers orange juice to milk, he wants a full glass of orange juice, but just a little milk in his glass. He may even choose a large glass for his juice and a small one for his milk. He measures temperatures by his comfort: the potato is too hot to eat, the ice is too cold to hold, he is too hot for a coat. The coming of night may denote bedtime; his hunger may denote mealtime.



As the child matures, the unstandardized units of measure which he commands increase in number and in separateness from himself. Also, he uses the number concepts which he is acquiring to define his measurements with increasing specificity. He perceives, for example, that this drawer is the same height as that drawer. This book, which he carries in one arm, is about as heavy as the two which he carries in the other arm. There is about twice as much playing room in the back yard as in the front yard. It is about as long from nap time until supper as from supper until bedtime.

Gradually, too, he gains some familiarity with standardized units and instruments of measurement. Mother uses a tape to measure fabric. Daddy has him back up to a wall and, by reading numerals on a yardstick, tells him that he is  $3\frac{1}{2}$  feet tall. The child steps on the scales and his doctor, by shifting weights and reading numerals, can tell him that he weighs 40 pounds. Mother uses a measuring cup when she cooks. Eight o'clock is set as bedtime; therefore, when the hands of Daddy's watch reach certain positions, it is time to go to bed.

Familiarity with standardized measures, although important, does not insure their understanding. If the child is to use standardized measures with the same degree of understanding with which he uses such comparatives as "big," "up," and "less," he must develop certain concepts.

The following brief history of the development of measures, which, in many respects, parallels the individual child's growth in ability to express comparisons, identifies many of these concepts.<sup>1</sup>

## HISTORICAL DEVELOPMENT

Throughout the ages, man has sought to measure, with increasing precision, an increasing number of quantities. His first uses of measure, as are the individual child's, were centered around himself, his possessions, and his simple needs. His measure of how much game to kill was an amount at least sufficient to satisfy his hunger. Provided he had food, the fullness or emptiness of his stomach regulated his mealtimes. If he wished to protect himself from the cold, he measured animal skins by his body dimensions to fit himself with clothing. If he wished to cover the entrance to his cave, he used the width and height of the opening as his units of measure.

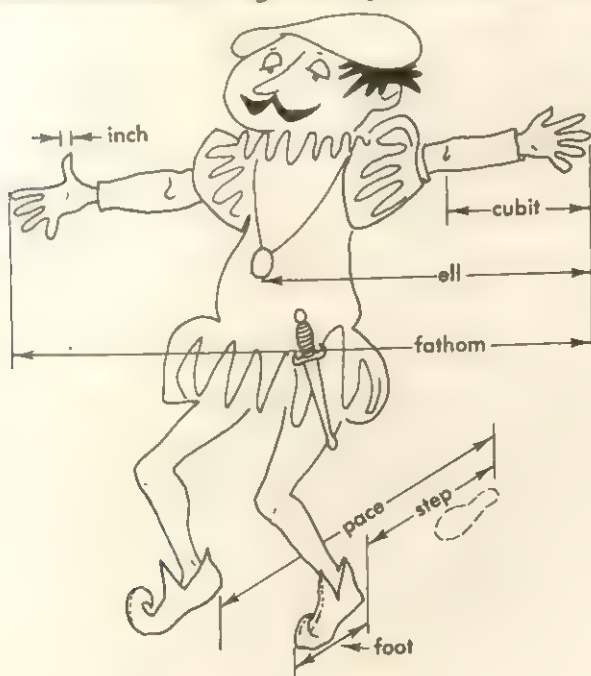
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<sup>1</sup> Another brief but interesting treatment of some aspects of the history of measurement can be found in the *New York Times* of June 8, 1958, VI, 48:4.

Long distances were measured by the arrow's flight or the number of days required to travel that distance. Such direct measures were not always convenient, however. For example, if the entrance to a man's cave was wide, he found it difficult to manipulate the large amounts of material for covering it during the entire measuring and fitting process. He discovered that by using certain parts of his body dimensions as units, he could first measure the opening, and duplicate this measurement on his materials without having to match the two directly until he had prepared his materials.

He used body units of varying sizes to suit differing purposes. For example, short lengths could be measured most simply by small units: his thumb's width, the breadth of his palm, his hand span; while longer units, such as his cubit (the length from elbow to finger tips), his ell (the distance from the center of his chest to the tips of his outstretched fingers), his fathom (the distance between finger tips with both arms outstretched), and his girth were used in measuring longer lengths. Distance on the ground was measured by the length of his foot, his step, and his pace (two steps).

When a man needed several pieces of material—for example, logs of the same length—he used the first one which was cut as a measuring instrument for the others. Such instruments often were cumbersome, and he devised instruments of lighter weight, such as small sticks, which



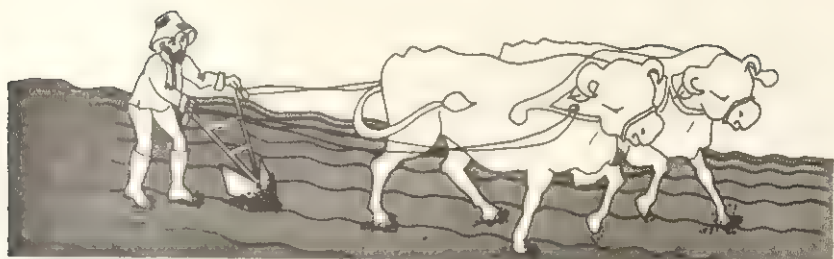
he cut and worked especially for measuring purposes. Experience with these instruments led him to sense a relationship between the various units. Two measurements with an ell stick for example, were about equivalent to one measurement with a fathom stick. Thus, he developed one stick marked with several units, as our yardstick is marked in inches.

These personalized units became increasingly inadequate as trade developed. A small man's handful was less than a large man's; a length of cloth measured in ells by a man with short arms was less than that measured by a merchant with long arms. At each market place some standards were developed.

With the invention of the balance, measurements of weight were made possible. Stones were used at the market place as standards of weight. Individuals duplicated these weights with stones of their own so that they could weigh their produce before taking it to market. It was found that if, for example, a stone's weight of grain could be contained in a basket of particular dimensions, the same basket when emptied and refilled, would contain an equivalent amount of grain (weight). As standard containers were put into use at the markets, men measured these containers (with linear units such as cubits, spans, etc.) and duplicated them for home use. Thus, the measures of capacity were derived from measures of weight and were often substituted for them.

As man found that things could be measured in an increasing number of ways, he also developed means for measuring an increasing number of quantities with greater precision. For example, carob seeds (from which our word "carat" comes) and grain were used as standards for weighing quantities too small or precious to be weighed





with stones. Pieces of metal were made which were equivalent in weight to certain numbers of grains. Even heavy quantities, then, could be weighed to the precision of the lightest grain-derived weight.

As men organized governments and levied taxes for their support, need for measurement of area arose. Large land areas were measured by the amount of grain needed to sow them and the number of animals needed to cultivate them. Measures of smaller areas were derived from the time taken to plow them. The acre, for example, was the amount of land which could be plowed in a morning's time.

Increasing mobility and wider trade made further arbitrary standardization necessary. Standards were often set by royal decree. The inch, originally established by the Romans as a twelfth of a foot measure, was declared by King David of Scotland to be the average width of the thumbs of a small, middle-sized, and large man. In England, King Edward II decreed that the inch was to be the length of 3 barleycorns, that 12 barleycorn inches were to make a foot, and that 3 such feet were to make a yard.

The first official pound weight used in England was kept in the Tower of London as a standard for making coins. Later, the Troy pound (from the Troyes market in France) was duplicated for use in England. Since there was usually some error in weighing heavy materials, it was thought that some compensation was needed; therefore, the avoirdupois (goods of weight) pound, heavier than the Troy pound, was established as the standard for weighing such commodities as fabrics and meat. The gallon, originally any vessel for holding liquids, was decreed by Henry III to be a container for holding 8 pounds of wine. Eight such gallons made a bushel.

As science and technology continued to develop, various changes were made in measuring standards to increase precision. In 1824 the

English government decided to base all standards of measure on a standard derived from the invariable length of the seconds pendulum on a clock. The inch was declared to be the length of one of the 39,1393 parts into which the pendulum's length could be divided; the yard, thirty-six of these. The weight of one cubic inch of water under controlled conditions became the base standard for weight. Standards of capacity were derived from these. Although superior to former ones, these standards were rather crude in comparison with those of the present. There still were at least two English gallon measures: the wine gallon, 231 cubic inches, and the ale gallon, 282 cubic inches. The metal weights and bars on which the length standards were marked were more subject to corrosion, expansion, and contraction than those of today.

The American colonists inherited this rather motley group of standards from the English. By 1850 some of the duplications (such as the ale gallon) had been eliminated, and copies of the standards kept in Washington were sent to the various states and to customs houses throughout the country.

The accompanying table, taken from a textbook<sup>2</sup> published in 1859, shows some units with which we are still familiar and some which have passed from common usage. At least one of the statements is incorrect—that four weeks make one month.

Meanwhile, people of other nations were also seeking to improve the accuracy and decrease the multiplicity of standards of weights and measures. Interest in developing international standards grew, and in 1875 the International Bureau of Weights and Measures was established. Subsequently, all member nations received copies of the meter length and kilogram weight chosen as international standards. The original is kept near Paris in a subterranean vault of a building of the International Bureau.

The present American standards, derived from these copies of the international meter and kilogram, were made legal by Congress in 1866 and were adopted as official standards by the Bureau of Standards in 1893. Our yard, for example, has been until quite recently  $3600/3937$  meter.

On July 1, 1959, the major English-speaking countries of the world began using a previously agreed upon uniform inch measurement. The new inch is 2.54 centimeters. This inch has been in use by United States industry since 1933. The United States Coast and Geodetic

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<sup>2</sup> E. F. Davis, *Practical Book-keeping and Arithmetic* (Philadelphia: J. B. Lippincott Co., 1859), pp. 43-44.

## WEIGHTS AND MEASURES

(1859)

*What are the denominations of Federal Money?*

10 mills	make	1 cent.	ct.
10 cents	make	1 dime.	d.
10 dimes	make	1 dollar.	\$
10 dollars	make	1 eagle	E.

*What are the denominations of Wine Measure?*

4 gills (gi.)	1 pint.	pt.
2 pints	1 quart.	qt.
4 quarts	1 gallon.	gal.
63 gallons	1 hogshead.	hhd.
2 hogsheads	1 pipe.	p.
2 pipes	1 tun.	T.
31½ gallons	1 barrel.	bbl.
42 gallons	1 tierce.	tier.
84 gallons	1 puncheon.	pun.

*What are the denominations of Sterling Money?*

4 farthings	1 penny.	d.
12 pence	1 shilling.	s.
20 shillings	1 pound.	
5 shillings	1 crown.	
21 shillings	1 guinea.	

*What are the denominations of Troy Weight?*

24 grains	1 pennyweight.	pwt.
20 pwt.	1 ounce.	oz.
12 ounces	1 pound.	lb.

*What are the denominations of Apothecaries' Weight?*

20 grains (gr.)	1 scruple.	
3 scruples	1 drachm.	
8 drachms	1 ounce.	
12 ounces	1 pound.	lb.

*What are the denominations of Avoirdupois Weight?*

16 drams (dr.)	1 ounce.	oz.
16 ounces	1 pound.	lb.
25 pounds	1 quarter.	qr.
4 quarters	1 hundredw't.	cwt.
20 hundredw't.	1 ton.	T.

*What are the denominations of Dry Measure?*

2 pints (pts.)	1 quart.	qt.
8 quarts	1 peck.	pk.
4 pecks	1 bushel.	bus.
36 bushels	1 chaldron.	ch.

*What are the denominations of Land or Square Measure?*

144 square inches	1 sq. foot.	ft.
9 square feet	1 sq. yard.	yd.
30¼ square yards or 272¼ sq. feet	1 sq. rod.	rd.
40 square rods	1 sq. rood.	r.
4 square roods	1 sq. acre.	A.
640 square acres	1 sq. mile.	M.

*What are the denominations of Cloth Measure?*

2¼ inches (in.)	1 nail.	na.
4 nails	1 quarter.	qr.
4 quarters	1 yard.	yd.
3 quarters	1 ell Flemish.	E. Fl.
5 quarters	1 ell English.	E. E.
6 quarters	1 ell French.	E. Fr.

*What are the denominations of Circular Measure?*

60 seconds (")	1 minute.	'
60 minutes	1 degree.	°
30 degrees	1 sign.	S.
90 degrees	1 quadrant.	
12 signs, or 360°	1 circle.	C.

*What are the denominations of Long Measure?*

3 barleycorns (b.c.)	1 inch.	in.
12 inches	1 foot.	ft.
3 feet	1 yard.	yd.
5½ yds.	1 rod, pole, or perch.	rd., p.
40 rods	1 furlong.	fur.
8 furlongs	1 mile.	M.
60 geographic or 69¼ statute miles	1 league.	
360 degrees, the circumference of the earth.		

WEIGHTS AND MEASURES—*Continued*

*What are the denominations of Solid or Cubic Measure?*

1728 solid inches	1 solid foot.	ft.
27 solid feet	1 solid yard.	yd.

40 feet of round timber or 50 feet of hewn timber

1 ton. T.

128 solid feet 1 cord wood. cd.

Note: A pile of wood 8 feet long, 4 feet wide, and 4 high, makes a cord because  $8 \times 4 \times 4$  is 128.

*What makes a dozen?*

12 single things	1 dozen.	doz.
12 dozen	1 gross.	gro.

*What makes a score?*

20 single things	1 score.
5 score	1 hundred.

*What are the denominations of Time?*

60 seconds	1 minute.	min.
60 minutes	1 hour.	ho.
24 hours	1 day.	da.
7 days	1 week.	wk.
4 weeks	1 month.	mo.
12 months or $365\frac{1}{4}$ days	1 year.	yr.
13 lunar months	1 year.	

*What makes a quire?*

24 sheets	1 quire.	qr.
20 quires	1 ream.	rm.

*What makes a hand?*

4 inches make 1 hand.

*What makes a fathom?*

6 feet make 1 fathom.

*What are the denominations of Ale or Beer Measure?*

2 pints	1 quart.	qr.
4 quarts	1 gallon.	gal.
36 gallons	1 barrel.	bbbl.
64 gallons	1 hogshead.	hhd.

Survey will still use 2.540005 cm. per inch in map making, though. In adopting this standard inch, the British have abandoned one of 2.539996 centimeters which was much more troublesome to use in problems of converting from one system to another.

The search continues for more precise standards, for new ways to apply these standards, and for new standards. For example, a new way of determining the standard for a meter, 100 times as precise as measuring the present platinum-iridium bar, has been found. This standard, derived by measuring one of the wave lengths of orange light emitted by krypton-86, is not only more precise but also easier to duplicate than the present standard. Similarly, the increase in our language of such terms as "acre feet," "60 gauge, 15 denier," and "passenger miles" indicates the progress being made in deriving new measures as civilization creates new needs for them.

## PROPERTIES OF MEASURABLE QUALITIES

Despite man's continuing discovery of ways by which measures may be applied, there are still more unmeasurable than measurable qualities around us. We have no precise methods, for example, of measuring such abstract qualities as goodness, love, hate, hope, and peace. Although great progress has been made by social scientists during the last century, these men would be the first to say that their measurements of intelligence, personality adjustment, etc. are indicative, rather than precise. Even the physical property of hardness cannot be measured with much success, due in part to the difficulty of defining hardness.

The qualities most amenable to mathematical treatment (and thus to precise expression) are those most resembling the real number scale. To expand and illustrate this statement, let us consider the comparative difficulties we might encounter in measuring (a) the length and (b) the tastiness of two identical chocolate candy bars. We know that in measuring the length of our candy bars the units are homogeneous, as are those of the number scale. But what of the units of tastiness? If any units could be named, they would probably be the flavors contributed by the various ingredients which went into the candy bars. Not only are these flavors heterogeneous (sugar and chocolate, for example, taste nothing alike), but the contribution that each flavoring agent made to the blended flavor of the candy bar could not be calculated with any exactness (one teaspoon of vanilla would have a much greater influence on the flavor of the candy bars than would one teaspoon of sugar).

The units of the ruler, like those of the real number scale, are in fixed serial order. If our ruler were sufficiently long, we could place the candy bars end to end to obtain their combined length. In measuring their tastiness, however, we have no fixed scale by which to go. We might take a poll among all our friends as to the tastiness of the candy bars, asking them to rate them "excellent," "good," "fair," and "poor," but again we have too many variables. One person's "good" rating might mean "about twice as good as fair," while another's might mean "just a little bit better than fair." One friend, a chocolate lover, might rate the bars "excellent," while another, disliking chocolate, might rate them "poor." And how do we place chocolate in an exact, fixed serial rating of tastiness? As yet, we have no satisfactory method. We can say that chocolate is one of the world's most popular flavors, that people buy more chocolate flavored confections than, say, licorice flavored ones; yet a person who preferred chocolate to licorice might still choose a "good" licorice candy over a "poor" chocolate candy.

Another property of the units of the ruler is that they are additive. We can measure the length of one of the candy bars, then of the other, and combine the two lengths to obtain a sum greater than the length of either, in fact, twice as great as that of either. Can we similarly state that if one candy bar tastes "fair," two candy bars taste "twice as fair" as one?

If the foregoing seems far removed from the understanding needed by the child, let us consider the child as a present and future consumer. He is constantly propagandized via radio, TV, and other communication media concerning the relative merits of almost every conceivable product. Some understanding of the properties of measurable qualities should be a useful part of his equipment for applying "the scientific attitude" to such claims.

## THE APPROXIMATE NATURE OF MEASUREMENTS

Although new measuring units and instruments are constantly being developed, all measurement is approximate. Limitations of exactness are inherent both in the measuring instrument and in our ability to read it. In Figure 2-1 we use a measuring instrument whose unit is the inch. Using this instrument to measure the length of line segments  $A$  and  $B$ , respectively, we term them each 3 inches long, because segment  $A$ , though less than 3 inches in length, is nearer 3 inches than 2 inches (more than  $2\frac{1}{2}$  inches); and line segment  $B$ , though more than 3 inches, is closer to 3 than to 4 inches (less than  $3\frac{1}{2}$  inches). We might ask how we know that  $A$  is nearer 3 inches than it is to 2 inches. This can be determined by comparing the length from 2 inches to the end of  $A$  with that from the end of  $A$  to 3 inches without any knowledge of the measurement in inches of either segment. We can say that our measurement of line segments  $A$  and  $B$ , respectively, is accurate to the precision of the smallest indicated unit, the inch.

Measuring  $A$  and  $B$  with the instrument in Figure 2-2, we now express the length of  $A$  as  $2\frac{1}{2}$  inches and that of  $B$  as  $3\frac{1}{2}$  inches,  $A$  being closer to  $2\frac{1}{2}$  inches than to 3 inches (less than  $2\frac{3}{4}$  inches), and  $B$  being closer to  $3\frac{1}{2}$  inches than to 3 inches (more than  $3\frac{1}{4}$  inches).

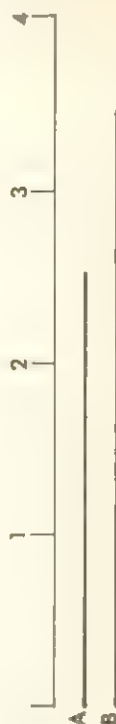


FIGURE 2-1



FIGURE 2-2



FIGURE 2-3

These expressions of measurement to the nearest  $\frac{1}{2}$  inch, we can see, still are inaccurate, but each is more precise than those in Figure 2-1.

Using the  $\frac{1}{4}$  inch units of Figure 2-3, we term line segment  $A$   $2\frac{2}{4}$  inches ( $2\frac{1}{2}$  inches), because it is closer to  $2\frac{1}{2}$  inches than to  $2\frac{3}{4}$  inches (less than  $2\frac{5}{8}$  inches). We term  $B$   $3\frac{2}{4}$  inches ( $3\frac{1}{2}$  inches), because it is closer to  $3\frac{1}{2}$  inches than to  $3\frac{3}{4}$  inches (more than  $3\frac{3}{8}$  inches). Although our instrument will measure only to the precision of the nearest  $\frac{1}{4}$  inch, we can still observe with the naked eye the inaccuracy of this measurement. Since we assume that any line segment, including segments  $A$  and  $B$ , can be divided into an infinite number of parts, even the smallest imaginable unit is still divisible; therefore, because it is

impossible for us to design an instrument of perfect precision, any expression of measurement contains some error, that is, some lack of precision.

Referring again to Figures 2-1 to 2-3, we see that the *greatest possible error* in measurement, the *tolerance*, is one-half the smallest named unit. In measuring line *A* to the precision of the inch unit (Figure 2-1), for example, we expressed the measurement as 3, rather than 2 inches, because we could see that it was more than half a unit (1 inch) longer than 2 inches, and thus closer to 3 inches than 2 inches. In this example, any error (and particularly the greatest possible one) in our measurement expressed as 3 inches falls within the  $\frac{1}{2}$  inch segment between  $2\frac{1}{2}$  and 3 inches. Similarly, the tolerance of our measurement of line segment *A* in Figure 3 is  $\frac{1}{8}$  inch, specifically the  $\frac{1}{8}$  inch segment between  $2\frac{1}{2}$  inches and  $2\frac{5}{8}$  inches.

The leeway for error of any expressed measurement, or the *tolerance interval*, is twice the tolerance of the smallest named unit. Again using Figure 2-1 to illustrate, we note that segment *A* is only slightly more than  $2\frac{1}{2}$  inches long, while segment *B* is almost  $3\frac{1}{2}$  inches long. Yet, though there is almost an inch (the unit) difference between the two, they are both termed 3 inches long: *A*, because its tolerance is within the  $\frac{1}{2}$  inch segment preceding 3 inches, and *B*, because its tolerance is within the  $\frac{1}{2}$  inch segment following 3 inches. Thus, any line segment terminating in the interval between  $2\frac{1}{2}$  inches and  $3\frac{1}{2}$  inches would be called 3 inches. Similarly, a measurement assigned the length of  $3\frac{1}{16}$  inches has a tolerance interval of  $\frac{1}{16}$  inch, specifically the interval between  $3\frac{1}{32}$  and  $3\frac{3}{32}$  inches.

The ratio of the tolerance to the total measurement is called the *relative error*. The significance of a  $\frac{1}{2}$  pound error in weighing a battleship would be inconsequential; of an elephant, slight; but of a small kitten, quite great. This is the reason we usually consider that

measurement which has the smallest relative error to be most precise. The relative error of our measurement of segment  $A$  in Figure 2-1 would be computed as  $\frac{1}{2} \div 3$  and expressed as  $\frac{1}{6}$ ; in Figure 2-2,  $\frac{1}{10}$ ; and in Figure 2-3,  $\frac{1}{20}$ . The relative error expressed as a percentage is termed the percentage of error (or merely percent error).

Although the elementary school child need not be concerned with the scientist's and mathematician's terminology of precision in measurement, he does have use and need for the following related understandings:

1. No measurement is exact.
2. Unless otherwise indicated, a measurement is considered precise to the smallest named unit.
3. The precision of the instrument is one of the factors which determines the degree of precision with which we can measure.
4. When the quantity being measured is held constant, the smaller the unit, the more precise the measurement; therefore, to obtain a more precise measurement of a given quantity, a more precise instrument is needed.
5. While large quantities can often be measured adequately with large units, smaller quantities generally require the use of small units for precision in measurement. In this sense we might say that when the unit is held constant, the greater the quantity being measured, the more precise the measurement.
6. The degree of precision required in a measuring instrument is determined by the degree of precision required in the measurement or, in other words, by the purpose for which the measurement is being made. Usually, despite inherent inaccuracy of all measuring devices, we can select instruments of sufficient precision for our various purposes.
7. Since all measurement is approximate, no expression of measured number can be considered unequivocal, as can a parallel statement about pure number (a number with no units associated with it).
8. In computations based on measurements we cannot be assured of accuracy any greater than the precision of the least precise measurement used. That is, no amount of computation can im-

prove the precision of a measurement. (In fact, computation may cause loss of precision.)

Statements 2, 7, and 8, which treat the expression of precision, need amplification. Few topics of elementary arithmetic have caused as much difficulty as this one.

For example, when we say that 3 plus 4 is 7, we can regard this statement as being true within the system in which the statement is made. When we say that 3 pounds and 4 pounds are 7 pounds, however, we must keep in mind that each term may express weight only to the nearest pound. If the weight expressed as 3 pounds is a measured quantity, which, more precisely, is near 3 pounds 7 ounces, and the weight given as 4 pounds is, more precisely, near 4 pounds 7 ounces, their combined weights, expressed 7 pounds 14 ounces, would actually be nearer 8 than 7 pounds. On the other hand, if their weights were more precisely termed 2 pounds 9 ounces, and 3 pounds 9 ounces, respectively, their combined weights, recorded 6 pounds 2 ounces, would be nearer 6 than 7 pounds. We see that we cannot know that the sum of two measured quantities, 3 pounds and 4 pounds, is correct to the nearest pound, because precision has been lost in the computation. Another measurement of these same objects to the accuracy of the nearest ounce would have a combined tolerance of one ounce, and, therefore, would be no less than 6 pounds 15 ounces, nor more than 7 pounds 1 ounce. Statements about measured quantities must always be considered to be approximate and can refer only to a particular measurement situation.

To further illustrate the approximate nature of computations based on measured quantities, let us consider the sum of two measured quantities: 3 lbs. 7 oz. + 4 lbs. It becomes immediately clear from the above discussion that we cannot hope to obtain a result correct to the nearest ounce. Hence, we "round" all terms to the precision of the least precise measurement. Even then, as implied in the paragraph above, we cannot know that the resultant, 3 lbs. + 4 lbs. = 7 lbs., is correct to the nearest pound.

As a last illustration of the problem of precision of results computed from measured quantities, let us consider  $13 \text{ lbs.} \div 4$ . If there were no errors in measurement, we could convert to 12 lbs. 16 oz.  $\div 4 = 3 \text{ lbs. } 4 \text{ oz.}$  This would imply that, in an approximate measurement situation,  $\frac{1}{4}$  of 13 lbs. is 3 lbs. 4 oz., correct to the nearest ounce, whereas, in actuality,  $\frac{1}{4}$  of 13 lbs. may be as small as 3 lbs. 2 oz., or as large as

3 lbs. 6 oz. The companion problem of multiplication is left to the reader, but it is sufficient to say that even more absurd results may occur.

## EQUIVALENT EXPRESSIONS OF THE SAME MEASURE

The same object can be measured in many ways. If we wished to measure a book, we could weigh it; determine its length, width, and depth; compute the area of its various surfaces and its volume; count its pages; measure the time needed for a particular person to read it; and even take its temperature. If, for example, we decided to measure its weight, the measurement could be expressed in milligrams, pounds, tons, etc. Children need to learn to choose instruments and units appropriate for their purposes in measuring, and to change an expression of measurement from one unit to other equivalent units. Through many experiences with several units of the same measure, they should form two generalizations: (1) in changing the expression of a measured quantity from larger to smaller units of measure, we multiply the number of larger units by the number of smaller units contained in one of the larger units, and (2) in changing the expression of a measured quantity from smaller units to larger units we divide the number of smaller units by the number of smaller units contained in one of the larger units.

## THE METRIC SYSTEM

We in the United States are so familiar with the common English measures (inches, pounds, quarts, etc.) that many of us are probably unaware that our official standards are metric. The metric system, developed by a committee of French scientists in 1799, has two advantages over the English system: first, conversion and computation are simplified because all units are derived from the base unit, the meter, by successively multiplying or dividing by ten; and second, the relationship of the various measures (of distance, area, volume, and weight), both within the same measure and between the measures, is evident from the names of the units. The following table shows the common metric measures.

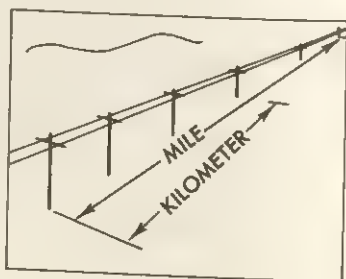
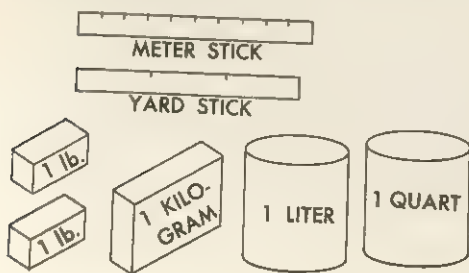
The original meter standard was based on a calculation of one ten-millionth of the distance from the equator to the pole. Measures of

area and volume are derived by squaring and cubing distance measures (square meter, area; cubic meter, volume). The gram, the base weight unit, is the weight of one cubic centimeter of water at four degrees centigrade. The capacity base, the liter, holds one cubic decimeter (more often expressed as 1000 cubic centimeters). The unit names most often encountered are meter, centimeter, kilometer, liter, gram, and kilogram.

	Thousands	Hundreds	Tens	Base Units	Tenths	Hundredths	Thousandths
<b>Distance</b>	kilometer	hectometer	dekameter	<b>Meter</b>	decimeter	centimeter	millimeter
<b>Weight</b>	kilogram	hectogram	decigram	<b>Gram</b>	decigram	centigram	milligram
<b>Capacity</b>	kiloliter	hectoliter	deciliter	<b>Liter</b>	deciliter	centiliter	milliliter

The metric system is conventionally used by scientists in this country as well as in those where it is in common daily use. We also encounter it in international sports events, such as the Olympic games. The English system continues in this country in everyday usage, however, not only because it is traditional, but also because its units, though diverse, are often more practical than those of the metric system. For example, there are no close metric approximations to our convenient foot or inch measures, the nearest being a decimeter, 3.9 inches, and a centimeter, 0.39 inches.

It seems probable that the dual use of the systems will continue. Although emphasis in the elementary school should reflect the wider use in this country of the English system, an acquaintance with the metric system is also desirable. Of particular importance would be some knowledge of approximate equivalences between the more common units of the two systems: for example, that a meter is a little more than a yard, that a kilometer is slightly more than one-half mile, that a kilogram is a little more than two pounds, and that a liter is a little more than a quart.



## USING MEASURING INSTRUMENTS

In order to deal effectively with measurement situations, children must develop certain understandings and certain skills. An understanding of the meaning of any measuring instrument's scale, for example, is basic to developing skill in obtaining measurements with that instrument. Thus, in telling time, the child must understand the units of the clock (hours, minutes, seconds), the relationship of the units, the functions of each hand, and the method of interpreting time from a particular reading. Similarly, in measuring distance, capacity, weight, and temperature, he must be able to choose and read the appropriate instrument and interpret the reading he obtains. In measuring area and volume, he must have, in addition to skills in linear measurement, the understandings and computational abilities which will enable him to use the respective formulas meaningfully.

In addition, the pupil needs understandings which enable him to use each instrument accurately. For instance, the accuracy of a measurement of body temperature taken orally is impaired when the temperature of the mouth has been raised or lowered by food or drink. The accuracy of the measurement of an individual's weight is influenced by the weight of his clothing and sometimes even by the manner in which he stands on the scales. The accuracy of tape measurements are decreased by crookedness or slackness in the tape or by failure to place the end of the tape at the beginning of the distance to be measured. In using a ruler to measure the width of a piece of paper, the measurement will be inaccurate unless the ruler is perpendicular to the length of the paper.

It is the responsibility of the teacher in the elementary school to give the child opportunities for experiences in measurement and to guide him in his understanding of the meanings, skills, and techniques required. There will be no other period of time in the child's life when he can so successfully integrate his knowledge of numbers with his day-to-day experiences with measurement.

## EXERCISES

1. Distance is sometimes measured by time. ("A day's journey" is an example.) Name three measures of distance in terms of time.
2. List three instances of indirect measurement not listed in the text.
3. Name five units of measure based on body dimensions and describe their quantity or length.

## 4. Convert these measurements as indicated:

(a) _____ tsp.	60 TBsp.	_____ cups
(b) _____ 1760 ft.	_____ yds.	_____ mi.
(c) _____ 1 gr. (A)	_____ oz.	_____ lb.
(d) _____ pt.	_____ qt.	3.7 gal.
(e) _____ gr. (A)	_____ oz.	3.5 lb.
(f) _____ ft.	3800 yd.	_____ mi.

5. Enlarge on the need for standardized units of measure.
6. Report on the development of new units of measure, their meanings, and how such measurements are made.
7. Describe in detail the construction and use of some measuring instrument. Include a discussion of the precision or accuracy of measurements made with this instrument and whether it measures directly or indirectly.
8. An ophthalmologist measures the internal pressure of the eye by placing a small pressure sensitive instrument (a tonometer) against the surface of the eye. He reads the pressure by reading the amount of deflection of an indicator needle. Is this an example of
  - (a) indirect measurement,
  - (b) derived measurement?
9. Define and explain the difference between tolerance and relative error. Give three examples of measurements, stating the measurement, the tolerance, and the relative error.
10. Suppose a recipe calls for  $3\frac{1}{2}$  cups of flour and 1 cup of milk. Show the effect of multiplying the measurements used in the recipe by 100. What is the maximum possible error in the result? What is the relative error of the original measurement? Of the final result?
11. Use a ruler to measure the width, length, and thickness of your textbook. State all measurements found. What is the tolerance for each measurement? What is the relative error for each measurement?
12. In exercise 11, what can be said about the precision of the measurements? The accuracy?

*Select the appropriate response to each of the following statements and explain the reason for your choice.*

13. Which of the following is an example of direct measurement?
  - (a) measuring mileage on an odometer.
  - (b) measuring time with an hour glass.
  - (c) measuring the length of a desk with a tape measure.
  - (d) measuring electricity with an ohmeter.

14. Which of the following is an example of indirect measurement?
- (a) measuring capacity with a quart jar.
  - (b) measuring distance with a surveyor's chain.
  - (c) measuring angles with a protractor.
  - (d) measuring weight with a bathroom scale.
15. State whether each of the following measurements is direct, indirect, or derived:
- (a) The weight of the contents of a No. 2½ can is stated to be 1 lb. 13 oz.
  - (b) The speed of a bandit's car is stated to have been 95 mi. per hr.
  - (c) The time for one orbit of a satellite is stated to be 96.4 min.
  - (d) The width of a piece of dress material is stated to be 42 in.
  - (e) The volume of a wheelbarrow is stated to be 3 cu. ft.

*Select the appropriate response to each of the following statements and explain the reason for your choice.*

16. In measurement situations, when the greatest possible error remains constant, the relative error
- (a) increases when the quantity being measured increases.
  - (b) increases when the quantity measured decreases.
  - (c) is not related to the size of the quantity being measured.
17. In measurement situations, when the quantity being measured remains the same, the relative error
- (a) increases when the greatest possible error increases.
  - (b) decreases when the greatest possible error increases.
  - (c) increases when the greatest possible error decreases.
  - (d) remains the same whether the greatest possible error increases or decreases.

## Extended Activities

1. List arguments for and against the exclusive use of the metric system.
2. Define accuracy and precision as related to measurement. How do they differ? (For reading on this question, see Schaaf, *Basic Concepts of Elementary Mathematics*, pp. 294-306.)
3. Discuss three sources of error in measurement. (See Schaaf, *Basic Concepts of Elementary Mathematics*, pp. 294-306.)
4. Discuss the merits of the following statement:  
The difference between capacity measure and volume measure resides in the difference in the methods by which such measurements are

made; capacity may be measured directly by the application of the unit of capacity measure to the quantity to be measured, but volume is a derived measure dependent, in the final analysis, upon other direct measures.

5. Find examples of unusual measuring instruments. Tell what they measure. State whether the measurement of the measurable quality is direct, indirect, or derived.
6. How many pecks are in a hen? (Compare our peck with the ancient Egyptian hen.)

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## Chapter 3

Concepts to be developed in this chapter are:

1. *Algebra is a generalization of arithmetic.*
2. *Many of these generalizations take place at the elementary school level.*

# 3

## Algebra as a Generalization of Arithmetic

Not long ago in a discussion of the teaching of elementary school mathematics, the leader of the discussion said, "Let  $a$ ,  $b$ , and  $c$  represent three natural numbers." Later, the leader was asked, "Why did you use algebra in talking about elementary school mathematics? We don't use algebra in the elementary school." Is algebra used in the elementary school? What is algebra, anyway? Should it be included in the elementary school? We shall look at some answers to these and other questions in this chapter.

### WHAT IS ALGEBRA?

If we were to ask this question of an elementary school teacher, a high school algebra student, and a professional mathematician, we would probably receive three different answers. We might get even a fourth answer from a high school algebra teacher.

Let us consider several definitions—or attempts at definition. First, the general conception held by many students and laymen is that algebra is a collection of symbols, definitions, operations, and rules of procedure (some of which seem only vaguely sensible) in which letters are used to stand for numbers. It is unfortunate, but true, that to many people this is what algebra actually is. It is also true that this definition has in it many elements which are contained in the most precise, modern definition.

A more modern definition might read: algebra is a body of undefined terms, symbols, definitions, axioms, and the theorems which follow from these definitions and axioms. In this definition no mention is made of the conception that algebra—whatever it is—deals with numbers. In fact, a prominent mathematician who has contributed greatly to the field of the foundations of geometry was asked recently if it would be possible in the strictest modern mathematical sense to distinguish between geometry and algebra. After some thought he replied that he believed it would not be possible to distinguish between them.

A third definition is taken from *Elements of Mathematics* by Banks. It differs from the one in the preceding paragraph primarily in breadth. Banks says, "As it is now conceived, algebra is the study of mathematical systems." He defines a mathematical system as, "... consisting of undefined terms, axioms, definitions, and theorems."<sup>1</sup>

All of these definitions miss the mark of what algebra should be for the elementary school child and teacher. For them it should be sufficient to say that *algebra is a generalization of arithmetic*. This concept is a bridge between the arithmetic the child knows and the most general concept of algebra—a bridge he may use with confidence.

## PRONUMERALS

Dr. Max Beberman has used a word which should be in the vocabulary of every teacher of mathematics. It is the word "pronomeral." Just as in grammar a pronoun is a word which stands for a noun, a pronomeral is a symbol standing for a numeral (which stands for a number); so in the final analysis, a pronomeral stands for a number, but in its written or spoken context it represents a numeral. At the same time, like a pronoun, a pronomeral is less definite than the numeral for which it stands. The statement, "He is going to town," is

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<sup>1</sup> J. Houston Banks, *Elements of Mathematics* (Boston: Allyn & Bacon, Inc., 1956), p. 124.

less definite than the statement, "John is going to town." In spite of the fact that the former statement may convey the same meaning as the latter, it allows the leeway for the reader or hearer to substitute some other name for that of "John." It is in this sense that letters are used to represent numbers—or, more immediately, numerals. In sentence structure it is uncommon to see the statement, "\_\_\_\_\_ is going to town." It is common to use a pronoun instead of a blank. In mathematics it is more common to phrase the question, "Two plus how much gives seven?" as " $2 + x = 7$ ?" than it is to phrase it as " $2 + \quad = 7$ ?" In the statement " $2 + x = 7$ " the  $x$  is a pronumeral. It enables us to state our problem more naturally and with an economy of symbolism. In this sense, pronumerals are merely generalizations of the numerals used in ordinary arithmetic.<sup>2</sup>

It is sufficient to say here that those who will accept the use of letters to represent numerals in symbolic statements about numbers (in the sense outlined above) will not find an elementary introduction to algebra difficult or confusing.

## GENERALIZATIONS CURRENTLY USED IN THE ELEMENTARY SCHOOL

We have all seen letters used to represent numerals in certain written statements about numerical situations. In the elementary school the most common use, and the one most likely to come to mind, is in the statement of formulas:  $A = l \times w$ , the formula for the area of a rectangle. Other formulas for areas and volumes will doubtless come to mind also.

Other generalizations of arithmetic—that is, algebraic techniques and procedures—are used almost unconsciously in the elementary classroom. For example, in the type of subtraction problem in which we ask, "How many more are needed?" the child gradually begins to sense a pattern (a generalization) from the few specific problems he works. The generalization of the problem may take the form: if  $a$  and  $b$  are given numbers, and if  $x$  is the symbol representing the answer to the question, "How much must be added to  $a$  to give  $b$ ?" then  $a + x = b$ . Then, through many experiences from which he can generalize, the child may find that  $x = b - a$ . This generalization, although

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<sup>2</sup> Those interested in this subject are urged to read: Max Beberman, "An Emerging Program of Secondary School Mathematics," *Inglis Lecture—1958* (Cambridge: Harvard University Press, 1958).

seldom stated, is actually made by those who fully understand subtraction. That is, the answer to the problem  $b - a$  is actually the solution of the linear equation  $a + x = b$ .

In a similar fashion, after experiencing many specific multiplication problems, the child approaches division. He finds that  $96 \div 16$  means a number (in this case 6) such that 16 times "this number" equals 96. How much simpler it would be to let  $Q$  (or  $x$ , or any other letter) stand for "this number" and phrase the problem as follows:  $Q = 96 \div 16$  means that  $16 \times Q = 96$ . This can be a meaningful statement to the child. He can see that the letter  $Q$  denotes a number and that if he finds the correct value for  $Q$  (the numeral representing this number) and substitutes it into the statement  $16 \times Q = 96$ , he makes this a true statement. He will see that all other values for which  $Q$  might stand make the statement false. All of these generalizations are already in the child's mind in some form. Algebraic notation in this case makes it more convenient for him to state the generalizations. The definition of division—an inverse of multiplication—is seen to be merely a special case of a certain type of generalized arithmetic problem—that problem called in algebra the solution of the linear equation  $aQ = b$ . Algebra is not limited to alphabetic symbols. Various symbols can be used, and squares, circles, triangles, etc. have been found to be of value to elementary pupils inasmuch as they provide a place to write a numeral which may complete the statement. Examples include:

$$\square + 7 = 10; \quad 3 \times \triangle = 12; \quad 27 \div \bigcirc = 3$$

Without pursuing the topic, we can state that the area and volume problems worked in elementary school actually are problems of arithmetic, in which generalized (algebraic) notation is used and in which certain generalized techniques simplify the stating and working of the problems. For example, when we asked, "How long is the side of a square whose area is 16 square inches?" we are asking the child to solve the problem  $x^2 = 16$  for  $x$ . The simplified notation of the algebraic statement of the problem is much superior to the verbal statement, and in the final analysis only represents what the child must think in order to work the problem.

The association of numbers with a number scale is still another generalization of arithmetic used in the elementary school. This association is probably first encountered in the use of a ruler or yardstick but is later extended to other measuring instruments and, finally, to graphical representation of data.

The use of an equals sign in arithmetic entails much more than merely the use of the symbol for equal. The concept of equality is itself a generalization from many observations. The associative, commutative, and distributive properties are just a few of the generalizations (within the realm of algebra) which are actually in use today in the elementary school classroom.

## ALGEBRA AND ELEMENTARY SCHOOL SCIENCE

A fertile field for uses of mathematics and its generalizations is elementary school science. Only in recent years has a real beginning been made in the introduction of science to the elementary school child. Here the generalizations of arithmetic can be used to enhance the child's understanding of both science and mathematics. To illustrate, we will consider selected examples.

The generalization that is involved in the introduction of the concept of negative number is algebraic in nature. Illustrations of the use of negative numbers may be found within the experience of the elementary school child. One such illustration is represented in the use of the thermometer. The thermometer has a scale which is a special case of a number scale. On this scale can be recorded both positive and negative readings of temperature, and hence both positive and negative numbers can be used to represent and communicate these facts.

Scientific facts and instruments can also be used to enhance mathematical understanding, as in the use of a balance scale. Many of the facts about equations may be illustrated by using the balance. For example, if equals are added to or subtracted from both sides of a balance scale which is in balance, no effect is shown by the balance. Or, if the scale is unbalanced, addition or subtraction of equals leaves it unbalanced in the same order. This is equivalent to operations on equations and inequalities.

$7 = 3 + 4$	$7 = 3 + 4$	$12 > 8$ (read 12 is greater than 8)
$\therefore 7 + 5 = 3 + 4 + 5$	$7 - 2 = 3 + 4 - 2$	$\therefore 12 - 3 > 8 - 3$

(The symbol  $\therefore$  means therefore.)

All of these equations and inequalities are algebraic statements in the sense that they are generalizations of arithmetic.

Deductions from observed data recorded during a science class may be expressed as a formula—using letters or pronumerals to stand for numbers. As an example, with a string and a weight the child may build a pendulum to be suspended from the ceiling or a tree limb. He can time several swings of the pendulum. His recordings of the times and number of swings can be analyzed to show that for a pendulum of fixed length, the time of one complete swing is independent of the size of the angle of the swing. Another child, with a pendulum of different length, finds that while the time for one swing (the period) of his pendulum does not depend on the size of the initial angle, it does differ from the period of a pendulum of different length. He can then be led to a consideration of the general law governing the relation between length and period. He doubtless would not discover empirically the law  $T = 2\pi \sqrt{\frac{L}{g}}$ , in which  $T$  is the period in

seconds,  $L$  is the length of the pendulum in feet, and  $g$  is the acceleration of gravity (approximately 32 ft. per sec. per sec.); but he might be led to a discovery that  $T^2 = kL$  in which  $k$  is a pure constant not depending on length, angle of swing, or any other observable factor. In fact  $k$  has an approximate numerical value of 1.2. The accompanying table of values was constructed from actual observations. The timing was done using the sweep second hand of an ordinary wrist watch. If a stop watch had been used, more accurate data could have been obtained. The important mathematical lessons which might be learned from this problem are linked to the data recording and analysis, to the formulation of a tentative conclusion (through observation and intuition), to the relation between the data (the formula), and finally to the verification of whether or not the data satisfy the tentative formula. While learning some mathematics, the child will simultaneously be experiencing an introduction to the true scientific method.

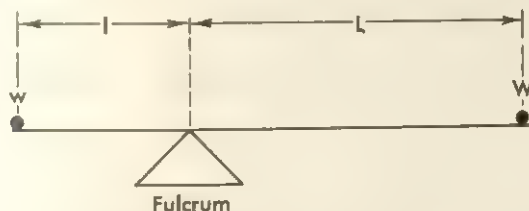
Another science experiment which can be performed and analyzed at the elementary school level is one using a lever and fulcrum. From his experiences with a seesaw the child is already familiar with the principles involved. From his observations he can be led to the conclusion that  $l \times w = L \times W$ . (See the accompanying diagram for symbol key.) Here again, letters are used to represent numbers, and an equation is used to express the relation between the various numbers. That is, some generalized arithmetic is being used.

In completing the consideration of science integrated with mathematics, it might be helpful to suggest that we use the following steps: (1) observe experiment, (2) record numerical data related to experiment, (3) generalize verbally from the data, (4) formalize the verbal

## 7 SWINGS OF A PENDULUM

LENGTH	TIME
$2\frac{1}{2}$ feet	13 sec.
4 feet	15 sec.
$6\frac{1}{2}$ feet	20 sec.

statement by the introduction of letters and symbols to obtain a formula or equation governing the behavior of the physical objects, and (5) verify by further observations that the data do behave according to the law formulated.



$w$  = number of pounds (or other weight unit)  
of first weight

$W$  = number of pounds (or other weight unit)  
of second weight

$l$  = number of feet (or other length unit)  
from fulcrum to first weight

$L$  = number of feet (or other length unit)  
from fulcrum to second weight

## A LOOK TO THE FUTURE

What aspects of algebra may we expect to be included in the elementary school curriculum in the future? Some of those to be mentioned here are already being introduced in experimental materials, some are already being used for enrichment, and some are already in

use in the regular classrooms of school systems using particular arithmetic series.

1. The four fundamental operations with all rational numbers may be introduced. This would include addition, subtraction, multiplication, and division of all signed numbers—integers and fractions.
2. The algebra of sets may be introduced and used, possibly only on an intuitive basis. This would include such things as set union, set intersection, the null set, the universal set (or universe), complements, etc., and the necessary symbolism to work with these new ideas.
3. New types of numbers not possessing all of the properties of ordinary arithmetic may be introduced. One example of these numbers and their properties is to be found in the study of matrices.
4. Finite arithmetics (arithmetic congruences, clock arithmetic) and their properties may be introduced.
5. An intuitive approach to the mathematical theory of limits is not beyond the comprehension of the elementary school child.

These five suggestions by no means exhaust the multitude of possibilities.

In closing this chapter, we urge the teacher of the elementary school child to keep an open and inquiring mind. The principles of the subject traditionally called algebra have their background in arithmetic. A few easy steps toward generalization of arithmetic on the parts of the child and the teacher will bring them to a vantage point from which great new vistas in mathematics and its applications will be opened to them.

## EXERCISES

1. Report on the association of number with a number line.
2. Show how a number line may be used to clarify rules for the addition and subtraction of both positive and negative numbers.
3. Find several definitions of "variable." Discuss their merits.
4. In physics we learn that for a convex lens, the focal distance  $f$ , the object distance  $p$ , and the image distance  $q$  are related by the equation (or law)  $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ . Determine the value of the unknown in each of

these situations. (This is an exercise in adding and subtracting fractions as well as an exercise in finding the reciprocal of a number.)

f	p	q
7 cm.	15 cm.	
3 in.		$3\frac{1}{3}$ in.
	15 ft.	2 in.
4 cm.	22.3 cm.	

5. Division is defined in terms of multiplication. Show how solving  $\frac{3}{4x} = \frac{2}{5}$  for  $x$  by the rules which are valid for treating an equation can lead to the proper result for  $\frac{2}{5} \div \frac{3}{4}$ .
6. Use letters to represent numbers in showing the validity of these two statements:
  - (a) If two fractions are equal, their reciprocals are equal;
  - (b) To add two fractions which have numerators of 1, put the sum of the denominators over their product.
7. Using the formula  $p = br$ , in which  $p$  stands for profit,  $b$  for base number (sales price) and  $r$  for rate, solve:
  - (a) A car sold for \$1600. If the profit was 25% of this price, how much was the profit?
  - (b) A car sold for \$1600. If the profit was \$400, what per cent of the sales price was the profit?
  - (c) If the profit on the sale of a car was \$400, and this is 25% of the sales price, what was the sales price?
8. Write an expression (formula) for:
  - (a) The cost  $C$  in cents of  $n$  seven-cent stamps.
  - (b) The cost  $C$  in dollars of  $n$  seven-cent stamps.
9. For what points on a number line will the number  $n$  associated with the points satisfy these relations?
  - (a)  $n \geq 1$
  - (b)  $n \neq 3$
  - (c)  $n \nlessgtr 5$  (the symbol  $\nlessgtr$  means "not greater than")

## Extended Activities

1. Think of a number from 1 through 9. Multiply it by 3. Add 2. Multiply by 3 again. Add the original number. The tens digit of the result is the original number. Why?
2. Multiplying any number by 25 can be accomplished by annexing two zeros and dividing by 4. Justify this statement by using  $n$  for the number and showing the relation between  $\frac{n}{25}$  and the quantity obtained by following the remainder of the rule.
3. Show that if  $\frac{a}{b} < \frac{a}{d}$ , then  $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ .

## Chapter 4

Concepts to be developed in this chapter are:

1. *There are arithmetics other than the one we customarily use.*
2. *The child has some experience with modular arithmetics.*
3. *Modular arithmetic offers students the opportunity to acquire new understandings of the roles of the fundamental operations and properties of arithmetic.*

## 4

## Introducing Modular Arithmetic

## A NEW ARITHMETIC

Suppose we were given the addition table in Figure 4-1. We might at first conclude that errors in type-setting were made. Upon closer examination, we would see that there is a definite pattern to the sums given in the table, and this pattern could hardly have occurred by accident.

For the moment, let us replace the symbol  $=$  by the symbol  $\rightarrow$ . This is done to reduce confusion about the meaning of the symbol. Examples of the facts given by this addition table are shown in the examples to the right of the table. It is to be noted that the numeral seven is not included in this arithmetic. It is not needed—as we will see shortly.

With the addition table as given, we can investigate the closure property of addition in this arithmetic. Is the sum of any two elements of this set also an element of the set? If so, the set is closed under

+	0	1	2	3	4	5	6	
0	0	1	2	3	4	5	6	$2 + 3 \rightarrow 5,$
1	1	2	3	4	5	6	0	$5 + 3 \rightarrow 1,$
2	2	3	4	5	6	0	1	$6 + 4 \rightarrow 3,$
3	3	4	5	6	0	1	2	$6 + 6 \rightarrow 5.$
4	4	5	6	0	1	2	3	
5	5	6	0	1	2	3	4	
6	6	0	1	2	3	4	5	

FIGURE 4-1

addition. The only elements of the set are 0, 1, 2, 3, 4, 5, and 6. The sum of any two of these elements is also one of these elements, as can be seen by the table. Therefore, the operation of addition is both closed and complete. Every pair of elements may be combined in the operation of addition by a mere reference to the given addition table.

Is it possible to define subtraction in such a system of arithmetic? The answer to this question is "yes." Subtraction is defined in the same way that it is defined in ordinary arithmetic. We say that  $a - b \rightarrow c$  if  $a \rightarrow b + c$ . Let us consider the example  $6 - 2$ . Does there exist a number  $c$  such that  $6 - 2 \rightarrow c$ ? If so, then  $6 \rightarrow 2 + c$ . By referring to the addition table, we see that  $c = 4$ , or  $6 - 2 \rightarrow 4$ . What of  $4 - 6$ ? If  $4 - 6 \rightarrow c$ , then,  $4 = 6 + c$ . By consulting the table we find that, in this problem,  $c = 5$ ; that is,  $4 - 6 \rightarrow 5$ . By examining all possible pairings of values of  $a - b$  and all possible resulting values of  $c$ , we find that the operation of subtraction is also closed and complete in this system of arithmetic.

Before proceeding, we should consider the question of whether or not there is an arithmetic system such as this. There actually are arithmetics in use which resemble this one, and, by a slight modification, we can find a use for this particular system.

Let us consider for a moment the following facts:

$$6 + 3 \rightarrow 9, \quad 7 + 5 \rightarrow 12, \quad 8 + 7 \rightarrow 3, \quad 9 + 11 \rightarrow 8.$$

Sunday	0
Monday	1
Tuesday	2
Wednesday	3
Thursday	4
Friday	5
Saturday	6

FIGURE 4-2

Are these facts compatible with any facts from the experience of the child or his teacher? You need look only as far as the clock face to find such an arithmetic as this.

What of the arithmetic first given in this chapter? Let the days of the week be numbered as in Figure 4-2. The problem of Wednesday + 6 days could be symbolized as  $3 + 6$ . The addition table given at the beginning of the chapter shows that  $3 + 6 \rightarrow 2$ . We find that 2 corresponds to Tuesday. Here, then, we find an illustration in our experience of the arithmetic with which the chapter began.

Arithmetic systems such as those described above are called "clock" arithmetic, modular arithmetic, or cyclic arithmetic. In these arithmetics, the sums and products, as we will see shortly, occur cyclically.

## MODULAR ARITHMETIC

In the case of the arithmetic for which the addition table was given at the outset, we can define a scheme which will generate this addition table. We will say that " $a$  is congruent to  $b$  modulo 7" if the difference  $a - b$  is divisible by 7 (that is, is divisible by 7 with a zero remainder). This is symbolized by " $a \equiv b \pmod{7}$ " and is frequently read in the abbreviated form " $a$  is congruent to  $b \pmod{7}$ ." In such a system  $6 + 5 = 11$ , but  $11 \equiv 4 \pmod{7}$ , since  $11 - 4$  is divisible by 7. If we replace the symbol  $\rightarrow$  by the symbol  $\equiv$ , we can verify the following facts and see that they correspond to those given earlier in the chapter.

$$2 + 3 \equiv 5 \pmod{7}$$

$$5 + 3 \equiv 1 \pmod{7}$$

$$6 + 4 \equiv 3 \pmod{7}$$

$$6 + 6 \equiv 5 \pmod{7}$$

In this modular arithmetic, any number is equivalent (congruent) to another number obtained by casting out any multiple of the modulus. Specifically, a number is congruent to a least non-negative number (residue) obtained by (1) casting out the modulus until the remaining

EXAMPLES	
(1)	(2)
$  \begin{array}{r}  33 \\  -7 \\  \hline  26 \\  -7 \\  \hline  19 \\  -7 \\  \hline  12 \\  -7 \\  \hline  5  \end{array}  $	$  \begin{array}{r}  4 \\  7 \overline{)33} \\  \underline{28} \\  5  \end{array}  $
	$33 \equiv 5 \pmod{7}$
	$33 \equiv 5 \pmod{7}$

FIGURE 4-3

number is one of the numbers 0, 1, 2, 3, 4, 5, 6, or (2) dividing by 7 and keeping only the remainder. The validity of each of the results in Figure 4-3 may be established by referring to the definition of congruences, since it is true that  $33 - 5$  is divisible by 7.

Using the definitions given above for congruent numbers, we can verify as in Figure 4-4 that one line of the addition table given at the outset may be obtained by the use of congruences and reduction of

$4 + 0 = 4$	$4 \equiv 4 \pmod{7}$	since $4 - 4$ is divisible by 7
$4 + 1 = 5$	$5 \equiv 5 \pmod{7}$	since $5 - 5$ is divisible by 7
$4 + 2 = 6$	$6 \equiv 6 \pmod{7}$	since $6 - 6$ is divisible by 7
$4 + 3 = 7$	$7 \equiv 0 \pmod{7}$	since $7 - 0$ is divisible by 7
$4 + 4 = 8$	$8 \equiv 1 \pmod{7}$	since $8 - 1$ is divisible by 7
$4 + 5 = 9$	$9 \equiv 2 \pmod{7}$	since $9 - 2$ is divisible by 7
$4 + 6 = 10$	$10 \equiv 3 \pmod{7}$	since $10 - 3$ is divisible by 7

FIGURE 4-4

sums to least non-negative remainders (residues). The verification of other lines of the table proceeds in the same way.

Let us now consider what multiplication means in such an arithmetic. If we define multiplication to be repeated addition, then,  $3 \times 6$  means  $6 + 6 + 6$ . In ordinary arithmetic  $6 + 6 + 6 = 18$ . In this modular arithmetic,  $18 \equiv 4 \pmod{7}$ , since  $18 - 4$  is divisible by 7. In modulo 7 arithmetic, then, we would say that  $3 \times 6 \equiv 4 \pmod{7}$ . The other entries in the following multiplication table (Figure 4-5) can be found similarly. It should be noted that, although each of the entries in the table is one of the numerals 0, 1, 2, 3, 4, 5, 6, the entries no longer occur cyclically in each row as they did in the addition table.

Is the operation of multiplication complete and closed? It is. Any two elements of the set 0, 1, 2, 3, 4, 5, 6 may be combined in the operation, and the product is always a member of the same set of numbers.

$\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

MULTIPLICATION TABLE  
Modulo 7

FIGURE 4-5

## DIVISION IN MODULAR ARITHMETIC

We now come to the problem of defining division in terms of this arithmetic. We will define division here just as we do in ordinary arithmetic—except that some odd consequences follow. We say that

$a \div b \equiv c \pmod{7}$  if there exists one and only one number  $c$  for which  $a \equiv bc \pmod{7}$ . We see that  $6 \div 2 \equiv 3 \pmod{7}$ , since  $6 = 2 \times 3 \pmod{7}$ . What about  $5 \div 2$ ? If  $5 \div 2 \equiv c \pmod{7}$ , then it is necessary that  $5 \equiv 2 \times c \pmod{7}$ . Referring to the multiplication table (Figure 4-5), we learn that  $c$  is 6. That is,  $2 \times 6 = 5 \pmod{7}$ . In fact, study of this multiplication table shows that each of the integers 0, 1, 2, 3, 4, 5, 6 occurs once and only once in each column of the table, and hence, that division of any integer by any other integer (except 0) is always possible in modulo 7 arithmetic.

We should understand that the definitions and techniques given above apply equally well to all other moduli. Only the results differ for other moduli. For the modulus  $M$ , it is true that:

- (1)  $a \equiv b \pmod{M}$  if  $a - b$  is divisible by  $M$ ;
- (2) the entries in each row of the addition table occur cyclically;
- (3) multiplication is defined in terms of repeated additions;
- (4) subtraction is defined in terms of addition;
- (5) division is defined (when a quotient exists) in terms of multiplication;
- (6) addition, subtraction, and multiplication are complete and closed operations.

However, it is not true that division is a closed and complete operation for every modulus. The multiplication table for modulus 6, which appears in Figure 4-6, illustrates this point. We would say, on

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

MULTIPLICATION TABLE  
Modulo 6

FIGURE 4-6

the basis of this table, that  $4 \div 5 \equiv 2 \pmod{6}$  since  $4 \equiv 5 \times 2 \pmod{6}$ , but no number  $c$  exists for which  $5 \div 3 \equiv c \pmod{6}$ , since there is no  $c$  for which  $5 \equiv 3 \times c \pmod{6}$ .

This last example serves to show that the operation of division is not complete in every modular arithmetic. That is, in modular arithmetic not every pair of integers may be paired in the operation of division. Hence, division is not an operation which is performed in modular arithmetic unless it is obvious that a quotient exists. This usually occurs only when the divisor is a factor (in the ordinary sense) of the dividend.

## OTHER PROPERTIES OF MODULAR ARITHMETIC

In ordinary arithmetic the symbol  $=$  stands for a relationship which possesses certain definite properties. Two of these properties could be paraphrased as "equals plus (or times) equals yield equals." Are these properties preserved in modular arithmetic? More explicitly, if  $a \equiv b \pmod{M}$  and  $c \equiv d \pmod{M}$ , is it true that  $a + c \equiv b + d \pmod{M}$  and that  $ac \equiv bd \pmod{M}$ ? To see that the answer is positive in each case, let us look back at the definition of  $a \equiv b \pmod{M}$ . This means that  $a - b$  is divisible by  $M$ , that is, that  $a - b = kM$  for some integer  $k$ . Similarly,  $c \equiv d \pmod{M}$  means that  $c - d = KM$  for some integer  $K$ . Since  $a - b = kM$  and  $c - d = KM$  are equations, we know that they may be added to obtain  $a - b + c - d = kM + KM$ . Using the commutative property on the left side of the equation, and the distributive property on the right side, we may write this as  $a + c - b - d = (k + K)M$ . Using both the associative and distributive properties on the left, we may, in turn, write this as  $(a + c) - (b + d) = (k + K)M$ . This last statement, though, shows us that the difference  $(a + c) - (b + d)$  is divisible by  $M$ , since it equals a multiple of  $M$ . Therefore, the definition of congruences assures us that  $a + c \equiv b + d \pmod{M}$ . This answers the question about whether congruences may be added as equalities are added.

It is only slightly more difficult to show that, if  $a \equiv b \pmod{M}$  and  $c \equiv d \pmod{M}$ , then  $ac \equiv bd \pmod{M}$ , for  $a \equiv b \pmod{M}$  means  $a - b = kM$  or  $a = kM + b$ , and  $c \equiv d \pmod{M}$  means  $c - d = KM$  or  $c = KM + d$  for some integer  $k$  and some integer  $K$ . Since  $a = kM + b$  and  $c = KM + d$  are equations, we know that we can multiply to obtain the following equations:

$$ac = (kM + b)(KM + d) = kKM^2 + kMd + KMb + bd.$$

Thus,  $ac - bd = (kKM + kd + Kb)M$ , and the difference,  $ac - bd$  is divisible by  $M$  (since it is a multiple of  $M$ ). Therefore, by definition  $ac \equiv bd \pmod{M}$ .

We can also ask whether commutativity, associativity, and distributivity are properties of modular arithmetics just as they are of ordinary arithmetic. These questions could be converted into questions about ordinary arithmetic if by the definition of congruences, we merely converted congruences into equations. Without providing the proofs, we can say that modular arithmetics possess the commutative, associative, and distributive properties just as ordinary arithmetic does.

The foregoing discussion has not been designed to give a complete treatment of modular arithmetic. It has been designed solely to give a brief glimpse of an arithmetic which differs from that with which we are all familiar but which displays many of the properties of ordinary arithmetic. Even this brief glimpse should afford some insight into the interdependence of definitions, properties (axioms), and results (theorems) of any arithmetic.

Many very beautiful and bizarre results are to be found in the study of modular arithmetic. Some of the most profound results in all of mathematics arise from these simple principles, many of which are within the grasp of the elementary school child.

## EXERCISES

1. By use of the definition show that:

(a)  $16 \equiv 7 \pmod{3}$

(c)  $129 \equiv 3 \pmod{9}$

(b)  $54 \equiv 28 \pmod{13}$

(d)  $-62 \equiv -46 \pmod{27}$

2. If  $16 \equiv 4 \pmod{3}$ , show that:

(a)  $16 \cdot 5 \equiv 4 \cdot 5 \pmod{3 \cdot 5}$

(b)  $16 \cdot 5 \equiv 4 \cdot 5 \pmod{3}$

3. Construct an addition table modulo 7.

4. Using the table of addition facts (modulo 7), do these additions and subtractions:

(a) 
$$\begin{array}{r} 4 \\ +5 \\ \hline \end{array}$$

(c) 
$$\begin{array}{r} 4 \\ -5 \\ \hline \end{array}$$

(e) 
$$\begin{array}{r} 7 \\ 4 \\ +6 \\ \hline \end{array}$$

(g) 
$$\begin{array}{r} 473 \\ 521 \\ +674 \\ \hline \end{array}$$

(b) 
$$\begin{array}{r} 14 \\ +5 \\ \hline \end{array}$$

(d) 
$$\begin{array}{r} 14 \\ -5 \\ \hline \end{array}$$

(f) 
$$\begin{array}{r} 57 \\ 34 \\ +26 \\ \hline \end{array}$$

5. Construct a multiplication table modulo 7.
6. Work these multiplication problems using the table of multiplication facts modulo 7:

$$\begin{array}{r} \text{(a)} \quad 6 \\ \times 3 \\ \hline \end{array}$$

$$\begin{array}{r} \text{(b)} \quad 16 \\ \times 3 \\ \hline \end{array}$$

$$\begin{array}{r} \text{(c)} \quad 15 \\ \times 24 \\ \hline \end{array}$$

7. In each part of Problem 6 revert to the definition of a congruence to show that the ordinary product is congruent modulo 9 to the product you found in Problem 6.

8. Construct a multiplication table modulo 8.

9. Multiply modulo 8.

$$\begin{array}{r} \text{(a)} \quad 6 \\ \times 8 \\ \hline \end{array}$$

$$\begin{array}{r} \text{(c)} \quad 24 \\ \times 7 \\ \hline \end{array}$$

$$\begin{array}{r} \text{(e)} \quad 78 \\ \times 64 \\ \hline \end{array}$$

$$\begin{array}{r} \text{(b)} \quad 7 \\ \times 4 \\ \hline \end{array}$$

$$\begin{array}{r} \text{(d)} \quad 36 \\ \times 51 \\ \hline \end{array}$$

10. Define the quotient  $a : b$  to be the integer  $c$  if (1)  $a \equiv bc \pmod{7}$  and (2)  $b \not\equiv 0 \pmod{7}$ .

(a) Does the quotient  $6 \div 3 \pmod{7}$  exist? What is it?

(b) Does the quotient  $6 \div 5 \pmod{7}$  exist? What is it?

11. Find the following quotients, modulo 7.

$$\text{(a)} \quad 5 \div 3$$

$$\text{(c)} \quad 0 \div 5$$

$$\text{(e)} \quad 1 \div 3$$

$$\text{(b)} \quad 4 \div 2$$

$$\text{(d)} \quad 3 \div 4$$

12. Repeat Problems 4, 6, and 11 using the modulus 11. This will necessitate construction of multiplication and addition tables for the modulus 11.

13. By definition,  $a \equiv b \pmod{m}$  if  $a - b$  is divisible by  $m$ . Show that this implies that if  $a$  is divided by  $m$  with non-negative remainder  $R_1$  and if  $b$  is divided by  $m$  with non-negative remainder  $R_2$ , then  $R_1 = R_2$  (provided both  $R_1$  and  $R_2$  are less than  $m$ ).

14. Verify by definition and by the results of Problem 13, that

$$\text{(a)} \quad 53 \equiv +117 \pmod{8}$$

$$\text{(b)} \quad 53 \equiv -117 \pmod{34}.$$

15. If  $a \equiv b \pmod{m}$  and  $c \equiv d \pmod{m}$ , show that  $a - c \equiv b - d \pmod{m}$ .

16. Show that any integer is congruent to the sum of its digits modulo 9.

17. Show that the following statement, using binary notation, is true.

$$01110 + 10011 \equiv 00001 \pmod{10^{101}}$$

(Note: Arithmetic of this kind is used in many modern computers.)

## Extended Activities

1. Place the names of the days of the week in order at equally spaced positions around the circle. Establish a one-to-one relation between these names and the natural numbers 1 to 7 taken in order. Show how the addition of any number of days to any day shown on the circle is equivalent arithmetic modulo 7. Show how a single hand like a clock hand can be rotated to show this arithmetic.
2. Prove that an integer is congruent to the alternate sum and difference of its digits modulo 11. To find this alternate sum and difference take the units digit, subtract the tens digit, add the hundreds digit, subtract the thousands digit, etc.

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## Chapter 5

Concepts to be developed in this chapter are:

1. *Geometric concepts begin with perceptions of physical realities.*
2. *Geometric concepts may be represented in a variety of ways.*
3. *Set ideas are useful in representing geometric concepts.*

## Understanding Geometry

The major portion of the mathematics curriculum in the elementary school consists of the study of numbers and number systems (arithmetic), and the study of space and spatial relationships (geometry). In the past, much attention was given to the study of numbers and number systems, but little attention was given to various aspects of geometry. Currently in elementary schools, more emphasis is being given to geometry, and it can be predicted that in the years ahead even greater emphasis will be given to this study of space and spatial relationships.

In the study of arithmetic, teachers are concerned with the tasks of extending the child's understanding of numbers through a widening range of experiences, and teaching him ways of recording the number aspects of these experiences. In geometry, the task is the same. The role of geometry instruction is the enhancement of the child's perception of his universe, and his methods of describing these perceptions.

## PRIMARY GRADES

The child comes to school with understanding of many geometric concepts. Usually he has some understanding of the nature of squares, triangles, and circles. He has spoken of lines in play and in his use of paper and crayons. Ideas of measurement, as related to a part of geometry, have been developing. He knows that his younger brother is smaller than he; that his father's car is larger than the lawn mower; and that the distance from his house to the store is less than the distance from his house to grandmother's house.

The primary grade program, which depends heavily on these early understandings, is designed to help the child comprehend *representations* of this real world he knows. These representations include both realistic pictures of the real world and abstract figures. The real world the child knows may be a circle of classmates in a playground game, a triangular shaped park near home, or a corral with an open gate. Figure 5-1 shows a picture of such a corral and a series of figures, one of which is an abstract representation of the corral. As a part of the discussion concerning the corral and the figure which represents it comes a consideration of curves, closed curves, and simple closed curves.

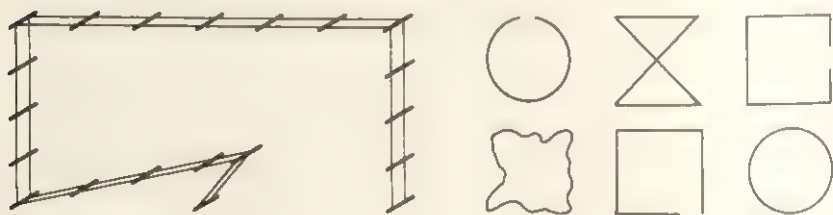


FIGURE 5-1

Particularly useful throughout the geometry curriculum is the peg-board, as shown in Figure 5-2, on which first grade children are able to construct and illustrate various simple closed curves.<sup>1</sup> (A curve is a set of points either "straight" or "curved" in the ordinary sense.)

Three-dimensional figures are also a part of the world of the primary grade child. Cubes, cones, spheres, and cylinders are familiar to him, and his understanding of these geometric figures is enhanced as he is helped

<sup>1</sup> See in particular: L. B. Smith, "Peg Board Geometry," *The Arithmetic Teacher*, XII (April 1965), 271-74; and "Geometry, Yes—But How?" *The Arithmetic Teacher*, XIV (February 1967), 84-89.

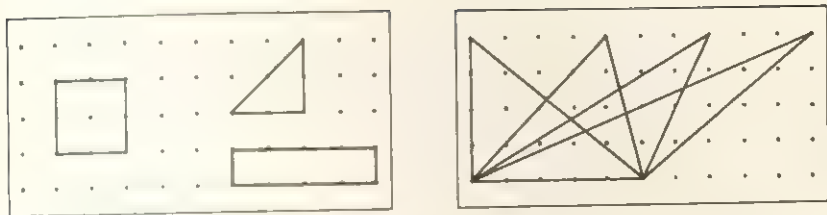


FIGURE 5-2

to bridge the gaps between his real world, realistic pictures of his real world, and abstract representations of this world. (Figure 5-3.)

By the time children are third graders they are ready for even more abstract consideration of geometry and are usually introduced to procedures and techniques for describing their real world in terms of sets of points. The universe is considered to be the set of all points. The concept of a point is used to identify locations, although points by themselves have neither size nor shape. Like numbers, points in the abstract do not exist physically, but are ideas in the minds of men. These are powerful ideas and are useful throughout our study of geometry.

A plane is thought of as a set of points which separates the universe of all points into two half-spaces. If a wall may be thought of as extending indefinitely, both left-to-right and top-to-bottom, this represents the idea of a plane. With such an idea of a plane, one imagines the set of all points separated into three sets of points: those points on the plane (those which are members of the set of points which compose the plane); those points on one side of the plane; and those points on the other side of the plane.

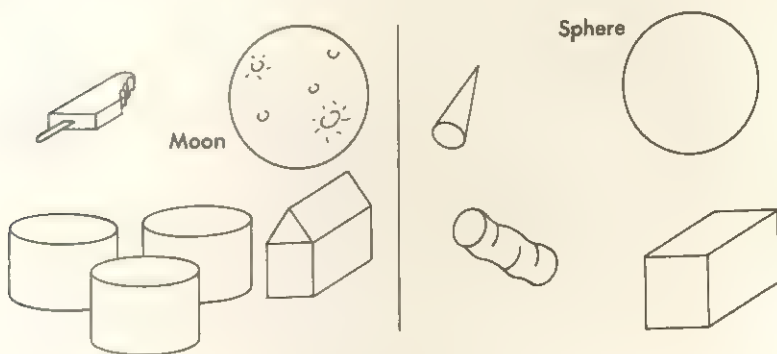


FIGURE 5-3

A line is also understood to be a set of points and if a point has no size and does not exist physically, then a set of these points as in a line, likewise, does not exist except in the mind. We do draw pictures to represent these ideas in geometry, however, just as we use numerals to represent number ideas in arithmetic.

A line is considered to be a set of points which extends indefinitely in both directions. In the elementary school the child considers lines which lie on given planes. A particular line can be identified by any two points on the line and is represented as shown in Figure 5-4.

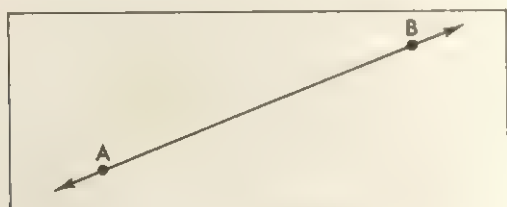


FIGURE 5-4

Line AB (denoted  $\overleftrightarrow{AB}$ ) divides the plane into three sets of points: those points on line AB; those points in the half-plane on one side of the line; and those points on the half-plane on the other side of the line. Particular sets of points on a plane are thought of as line segments: rays, angles, curves, polygons, and many other geometric figures. A line segment is a part of a line including two end points and all points between and is identified by its two end points. A segment is denoted, for example, as  $\overline{CD}$ . A ray is thought of roughly as half a line, including one end point, or as a part of a line extending indefinitely in a single direction as shown in Figure 5-5. As an example, notation for a ray is  $\overrightarrow{EF}$ . A point is represented simply by a dot. With these ideas represented, various geometric figures can be drawn to represent aspects of the physical world in which we live.

*Curves.* While it is difficult to define a curve in the abstract sense, and such a definition would not be useful here, it is not difficult to describe our representations of curves. They are continuous—can be traced with a pencil point without lifting the point from the paper—between their end points; and while they have length, they have no

width. In this concept, the word "curve" has no connotation of "curved" in the ordinary sense.

A polygon is a simple closed curve consisting of the union of three or more line segments: a triangle, three; a quadrilateral, four; a pentagon,

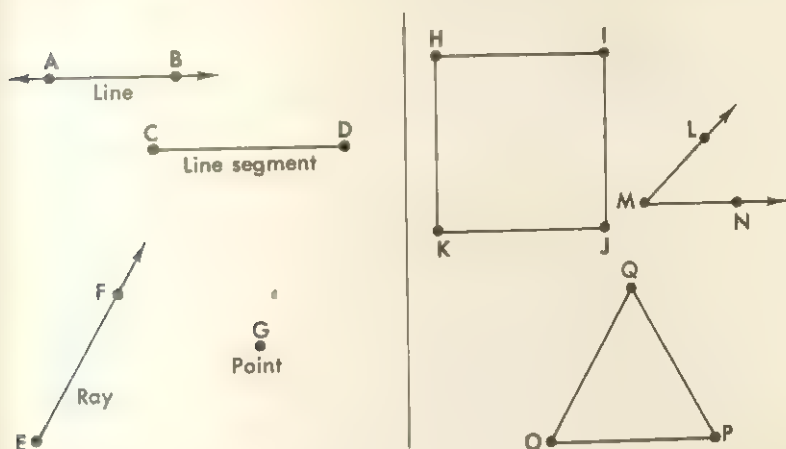


FIGURE 5-5

five; etc. Just as a polygon may be thought of as the union of three or more sets of points, i.e., three or more line segments, so an angle is the union of two sets of points or two rays with a common end-point called the vertex of the angle.

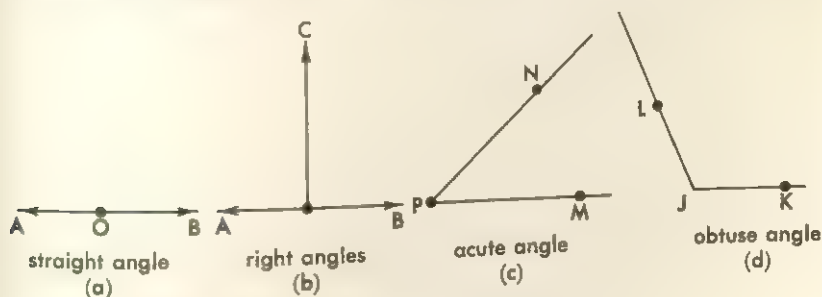


FIGURE 5-6

If the two rays forming an angle constitute a straight line, as illustrated in Figure 5-6(a), the angle is called a straight angle. If another

ray is perpendicular to the straight line at the vertex of the straight angle, it forms two adjacent angles which can be matched with each other. These angles are called right angles and are illustrated in Figure 5-6(b). It is conventional to define the measure of a right angle to be  $90^\circ$ . The degree measure of an acute angle, Figure 5-6(c), is less than  $90^\circ$ , while that of an obtuse angle, Figure 5-6(d) is greater than  $90^\circ$ .

The protractor in Figure 5-7 is shown placed along the line  $AF$  for the purpose of measuring the several angles shown in the figure. Which angle is a right angle? Why? Which angles are acute? Which are obtuse? Why?

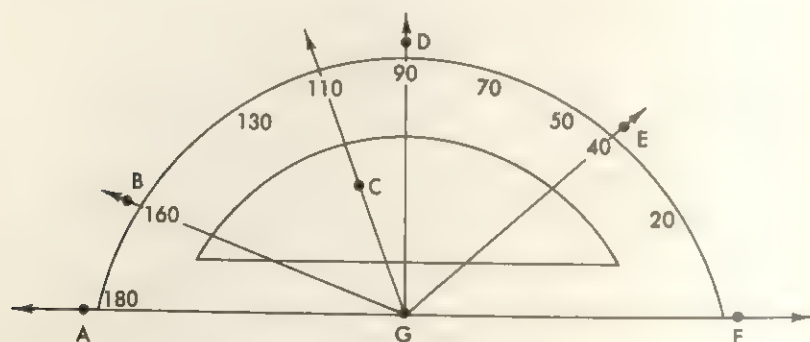


FIGURE 5-7

## INTERMEDIATE GRADES

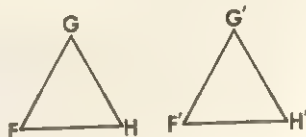
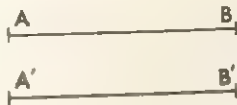
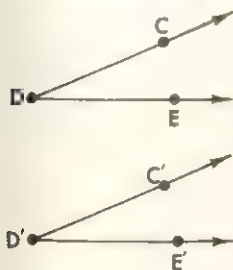
In the intermediate grades, geometry becomes increasingly formalized in abstract representations of the real world. The teacher provides experiences which increase the child's potential to communicate about the abstractions he is using. Symbol representations for rays, lines, line segments, points, and angles are more extensively employed in the child's descriptions of the physical world. At the same time, the child's understanding of sets and set ideas becomes useful as he discusses sets of points. A number of questions may be asked about the geometric figure represented in Figure 5-8. Is line segment  $EF$  a subset of line  $AB$ ? Is the union of ray  $AF$  and ray  $FG$  angle  $AFG$ ? Is the intersection of line  $AB$  and line segment  $CD$  point  $E$ ? Stated more simply, each of these questions may be represented as follows. Is each statement about the figure true?

1.  $\overleftrightarrow{EF} \subset \overleftrightarrow{AB}$  <sup>2</sup>
2.  $\overrightarrow{AF} \cup \overrightarrow{FG} = \angle AFG$
3.  $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = E$



• FIGURE 5-8

The concept of congruence is introduced in geometry. Intuitively, two geometry figures can be said to be congruent if: a) they have the same size and shape, or b) their parts can be matched. Line segment  $\overline{AB}$  is congruent to segment  $\overline{A'B'}$ , because they are the same length. Angle  $CDE$  is congruent to angle  $C'D'E'$  because the angles are the same size. We say that  $\triangle FGH \cong \triangle F'G'H'$ ,  $\overline{AB} \cong \overline{A'B'}$ . We do not say that  $\overline{AB} = \overline{A'B'}$  because  $\overline{AB}$  and  $\overline{A'B'}$  are not the same line. They are congruent lines because they can be matched exactly.

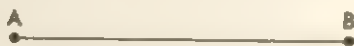


We can show congruency for these line segments as an equivalence relation demonstrating the three essential properties:

<sup>2</sup> The notation for subset ( $\subset$ ), union ( $\cup$ ), and intersection ( $\cap$ ) is familiar to the reader who has some knowledge of sets. Those who are unfamiliar with such notation may see Roger Osborn, M. Vere DeVault, Claude C. Boyd, and W. Robert Houston, *Understanding the Number System* (Columbus, Ohio: Charles E. Merrill Publishing Co., 1968), Chapter 2.

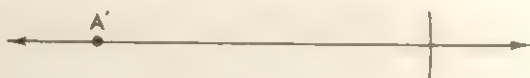
- $\overline{AB} \cong \overline{AB}$  (Reflexive property)
- If  $\overline{AB} \cong \overline{A'B'}$ , then  $\overline{A'B'} \cong \overline{AB}$  (Symmetric property)
- If  $\overline{AB} \cong \overline{A'B'}$  and  $\overline{A'B'} \cong \overline{A''B''}$ , then  $\overline{AB} \cong \overline{A''B''}$  (Transitive property)

Construction geometry often consists of replicating congruent figures. Straight-edge and compass construction activities can clarify many

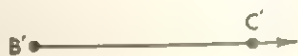
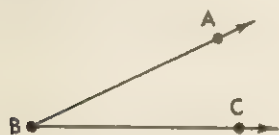


geometric concepts and, at the same time, increase the child's interest in geometry.

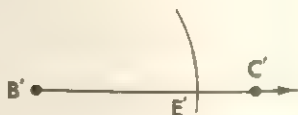
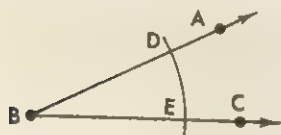
A line segment congruent to  $\overline{AB}$  can be readily constructed by drawing a line as in  $A'B'$ , placing a point  $A'$ , and then spreading the points



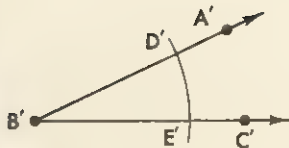
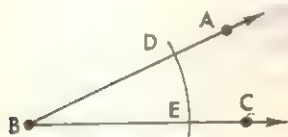
of the compass to reach from  $A$  to  $B$ . The point of the compass is then placed on  $A'$  and an arc is drawn indicating the point which should be labeled  $B'$ .



To construct congruent angles one begins in a similar manner: by constructing a ray  $B'C'$ . Next use the compass to draw an arc that intersects both rays of  $\angle ABC$ . Name the points of intersection  $D$  and

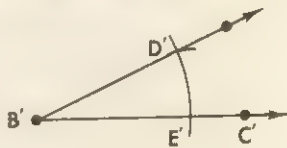
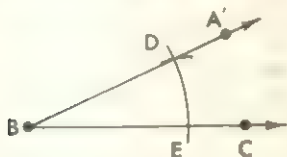


$E$ . Without changing the setting, put the point of the compass on  $B'$  and draw an arc intersecting  $B'C'$  at  $E'$ . Now use the compass to measure the distance between point  $E$  and point  $D$ . Place the point at  $E'$  and



draw an arc which determines point  $D'$ . Finally, draw a ray from  $B'$  through point  $D'$ . Thus, we have constructed  $\angle A'B'C' \cong \angle ABC$ .

Many activities involve the construction of a perpendicular bisector of a line segment. The manner of doing this can best be shown with the use of two intersecting congruent circles. The line segment between



the centers of the two circles is perpendicular to the line segment drawn between the two points of intersection of the two circles. Typically, perpendiculars are constructed without drawing the complete circles but are done in the manner in which  $\overline{A'B'}$  is intersected in Figure 5-9. The distance between  $B'$  and  $C'$  is the radius of one circle, and the distance between  $A'$  and  $C'$  is the radius of the second circle. These radii are equal because the same compass setting is used in the construction of each circle.

Increasing attention is given to the study of three-dimensional figures. Still keeping in mind the child's need to recognize geometry as a study of the space relationships in the real world, both pictures and abstract

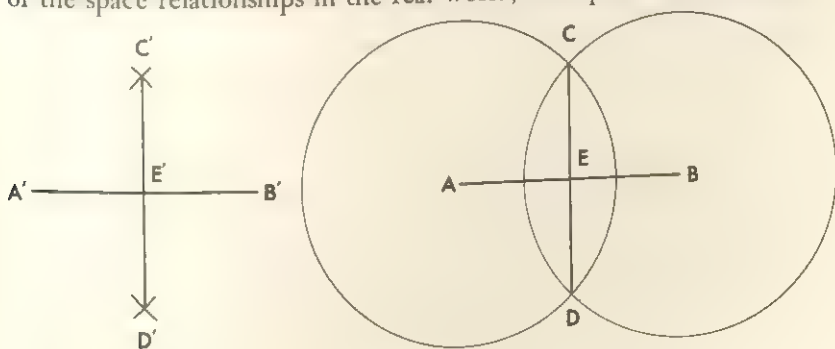


FIGURE 5-9

representations are used. In Figure 5-10, representations, pictorial and abstract, of a classroom are shown.

The child may think of the classroom as made up of the union of sets of planes. He learns to recognize that the intersection of two planes is sometimes a line, or, in the case of the classroom, the intersection of portions of planes may be line segments. The intersection of plane figures  $C B F G$  and  $F G H E$  is line segment  $F G$ . What set of points is the intersection of any two non-parallel planes? What set of points is the intersection of two parallel planes? What set of points is the intersection of three non-parallel planes? And with such queries the child's quest for understanding of the physical world in which he lives continues to expand.

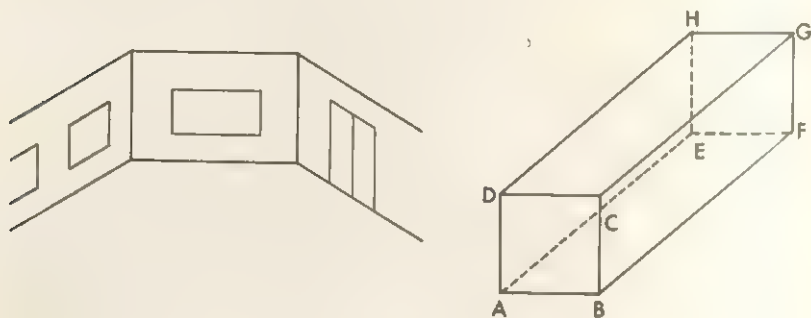


FIGURE 5-10

Metric geometry (measurement) is an important aspect of the curriculum throughout the grades and begins with the child's early experiences with the number line. Work with linear and volume measures also occupies much of the child's attention. Increasingly, elementary school programs are including coordinate geometry. Although some experimental work is being done with the use of coordinate geometry ideas in the early primary grades, it is usually included in the latter years of the elementary school program. The early use of tables and graphs provides excellent experiences which may be organized to establish a basis for more formal graphing experiences.

The child who organizes a bar graph to represent the data in the table in Figure 5-11 maps ordered pairs on a grid.

Relating the graphing of ordered pairs to the familiar task of identifying locations on a map has also been found to be a useful technique

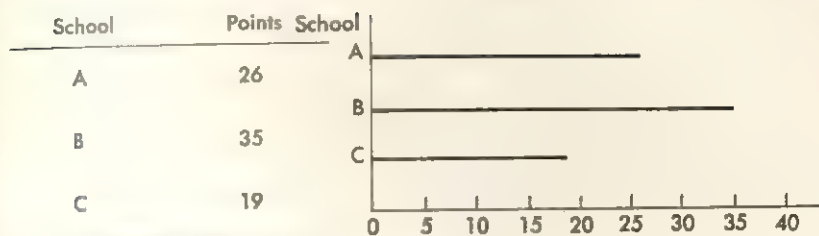


FIGURE 5-11

for the introduction of coordinate geometry. Figure 5-12 illustrates both the map and a coordinate grid on which ordered pairs are mapped.

First experiences as represented in Figure 5-12 include the child's locating intersections of avenues and streets on a map. He locates the intersection or the crossing of 2nd Avenue and First Street; 4th Avenue and Second Street; 3rd Avenue and Third Street; etc. Thus, even in

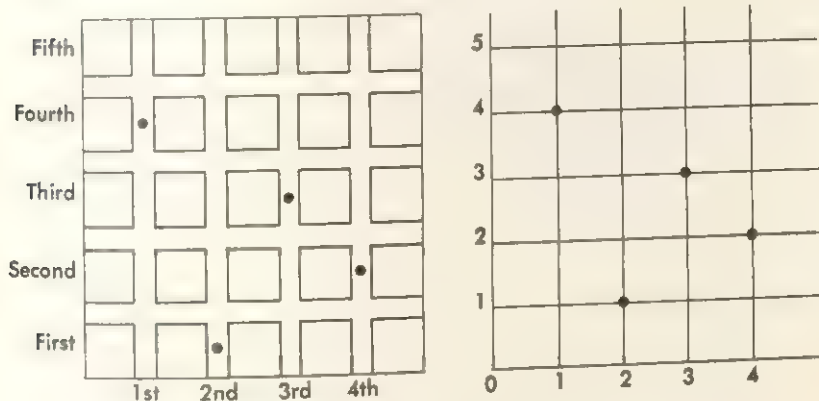


FIGURE 5-12

very early grades, it is relatively easy for the child to extend these understandings to the use of the coordinate grid, which is an abstract representation of the map, and to locate these points identified by the ordered pairs:  $(2, 1)$ ;  $(4, 2)$ ;  $(3, 3)$ ; and  $(1, 4)$ . It is conventional for the first element of the ordered pair to represent the horizontal displacement of the point from the  $(0, 0)$  point and the second element of the pair to represent the vertical displacement.

This grid may be extended to include all points on a two-dimensional

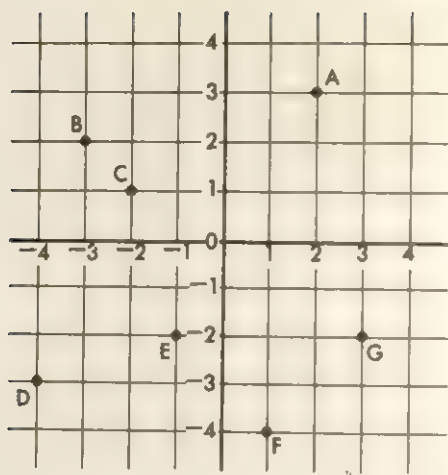


FIGURE 5-13

plane as pupils locate points identified by ordered pairs of integers. Such an extension of the coordinate grid is shown in Figure 5-13. Points A to F are identified, respectively, by the ordered pairs:  $(2, 3)$ ;  $(-3, 2)$ ;  $(-2, 1)$ ;  $(-4, -3)$ ;  $(-1, -2)$ ;  $(1, -4)$ ; and  $(3, -2)$ .

Graphing sets of ordered pairs is also included in many elementary school programs. The points graphed are frequently joined by a curve

3 + □ = △	
□	△
-4	-1
-3	0
-2	1
-1	2
0	3
1	4

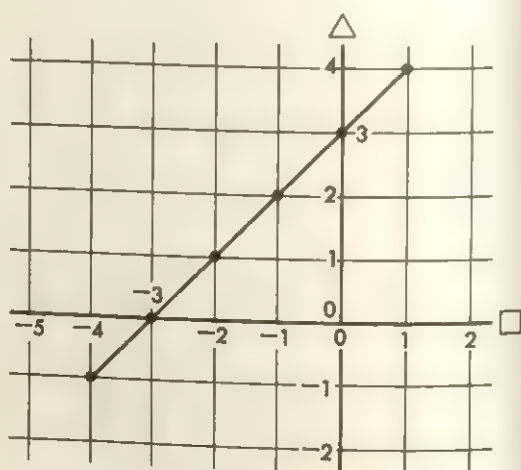


FIGURE 5-14

or by line segments from left to right. When equations with two unknown numbers are solved, they result in a set of ordered pairs which may be graphed on a coordinate grid as shown in Figure 5-14. Early instruction with coordinate grids often includes the use of open frames rather than  $x$  and  $y$  to identify the axes.

## SUMMARY

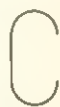
The Committee on the Undergraduate Program in Mathematics<sup>3</sup> has been influential in the promotion of increased mathematics requirements for prospective elementary school teachers. Significantly, its recommendations included substantial study of geometry. This chapter gives an indication of those geometry topics currently included in elementary curriculums. Some information has been provided for the practicing teacher who knows very little about geometry but wishes some understanding of the geometry included in the text program he is currently using. It has been developed also with the hope that geometry may appear not as the formidable high school topic which it seemed to many, but rather as a topic of considerable interest worthy of further exploration through suggested readings identified in the bibliography.

## EXERCISES

1. a. Which three of the following illustrations are *curves*?



a



b



c



d



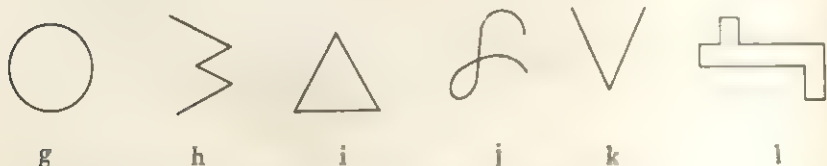
e



f

<sup>3</sup> The Committee on the Undergraduate Program in Mathematics (CUPM) is a committee of the Mathematical Association of America which has been charged with the task of preparing recommendations for improved undergraduate curricula of various types in colleges and universities. While the recommendations prepared have not been given the status of having been adopted by any accrediting agency, they have served as both criteria and guidelines in the revision of mathematics curricula at the college level.

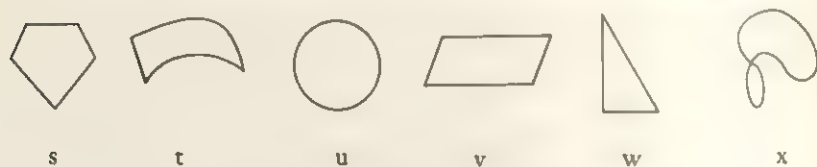
b. Which of the following curves are *closed curves*?



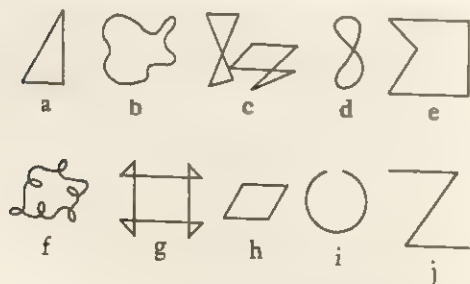
c. Which of the following are *simple closed curves*?



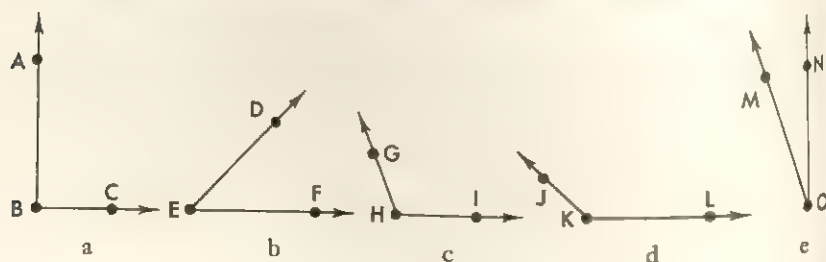
d. Which of the following simple closed curves are *polygons*?



2. Which of the following are simple closed curves?

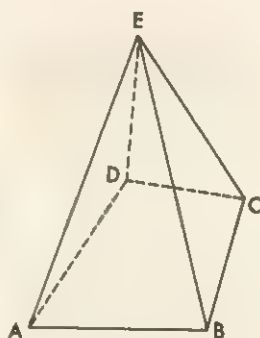


3. Which of the following angles are obtuse angles? Name the others.

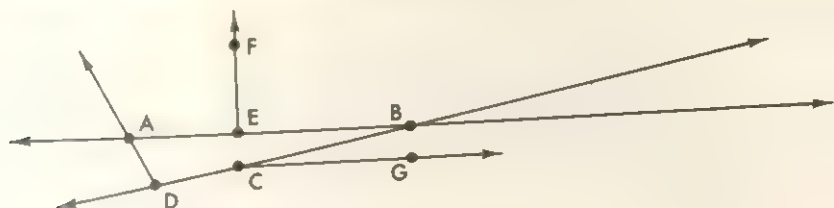


4. Considering the figure of the pyramid, determine which of the following statements are true.

- $\overline{EA} \cup \overline{AB} = \angle AEB$
- $\angle EAB \cap \angle CED = \text{point } E$
- $\square ABCD \cup \triangle EBC = \overline{BC}$
- $\square ABCD \cap \triangle EAD = \overline{AD}$



5. Complete each of the statements about the figure.



- $\overline{AE} \cup \overline{EB} = \square$
- $\overrightarrow{EF} \cup \overrightarrow{EB} = \square$
- $\overrightarrow{BA} \cup \overrightarrow{BC} = \square$
- $\overline{AB} \cup \overline{DA} \cup \overline{DB} = \square$
- $\overline{BE} \cap \overline{EF} = \square$
- $\overrightarrow{BA} \cap \overrightarrow{EB} = \square$
- $\angle BDA \cap \overrightarrow{CG} = \square$
- $\overleftrightarrow{AB} \cap \overline{AB} = \square$

6. Construct a coordinate grid and locate points for each set of ordered pairs.

- $(2, 4); (3, 1); (0, 5); (1, 4); (0, 0)$
- $(2, 3); (-3, 2); (-5, 1); (3, 5); (-1, 0)$
- $(-2, -4); (3, 5); (3, -2); (-3, 2); (-3, -1)$

7. Construct tables of values and coordinate grids and locate enough points to draw the curve described by each equation.

- $1 + \square = \triangle$
- $3 - \square = \triangle$
- $-2 + x = y$

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## Chapter 6

Concepts to be developed in this chapter are:

1. *Inductive reasoning utilizes statistical processes.*
2. *Quantitative data may be organized in graphs, tables, and similar forms for interpretation.*

# 6

## Using Statistics

Let us begin our dealing with statistics in this chapter by asking two questions. The first question is: "In a class of children, will there necessarily be a child who is exactly as tall as the average height of all of the children in the class?" The second question arises from a statement made on a television presentation of the world population problem. The commentator made the statement that "One out of every seven people on earth is an Indian." Suppose that you are not an Indian and that you and five of your non-Indian friends are gathered in a room. The second question now is, "Will the next person who enters the room be an Indian?" The commentator certainly said that every seventh person *is* an Indian. We need to look more closely at his statement and see what he meant to say.

### STATISTICS IN THE ELEMENTARY GRADES

Many statistical processes, techniques, and understandings are well within the grasp of the elementary school child. Since we more and

more frequently try to communicate with each other by associating numbers with physical phenomena, at an early age the child should be introduced to those phases of statistics which are within his grasp. It seems relatively certain that our approach to statistics in the elementary grades should center around experiences in the child's ever expanding environment, and that his activities should include collection, organization, and interpretation of data, and the more sophisticated arts of inference and decision-making.

### COLLECTING, ORGANIZING, AND INTERPRETING DATA

Contrary to what many textbooks would lead us to believe, the child's first introduction to the statistical process should center around the collection of data. Canned or prefabricated data have little meaning to the child. The collecting activity, furthermore, is one at which the child excels—as we can observe in the collections of bugs, dolls, marbles, play costumes, and just plain junk that children make. The collecting activities may be centered around any of many learning situations. Some examples of types of data to be collected are given here. The list is merely a scratch on the surface of possibilities, but is given in such a way as to illustrate relationships between the collection (and interpretation) of data and other fields of study.

1. Take a map of the school district. Divide it by north-south and east-west lines passing through the position of the school on the map. Find how many children in the classroom (or school) live in each of the quadrants.
2. Find how many blocks each child in the room (or school) walks or rides to get to school.
3. Find the number of square miles in each of the fifty states.
4. Find the height and weight of each child in the room.
5. Find the number of hours each child spends sleeping, playing outdoors, watching TV, etc.
6. Find the number of new words each child has correctly spelled each week.
7. Find from the weather bureau or newspaper files the rainfall totals for each month of the preceding year.
8. Record the number of games won by each team in the room.

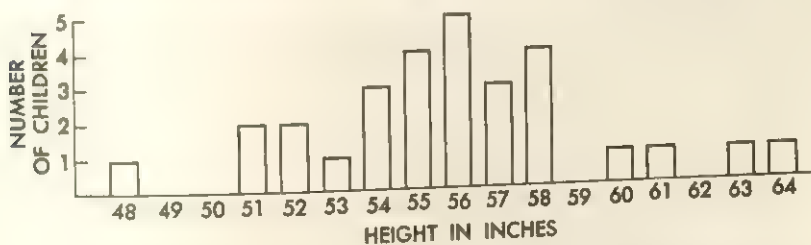
Having collected some data, the child will almost surely recognize that the raw data are useless for most purposes unless they are correlated or organized so that they may be interpreted. The child may need

help in the organizational activities. First, he will have to decide—or be led to a decision—about what sort of organization will be meaningful. Rainfall amounts may merely be totaled, but the total height of all children in a room would be, in itself, quite meaningless unless an acrobatic team were being organized. Usually organization centers around some graphic or tabular display of the information obtained.

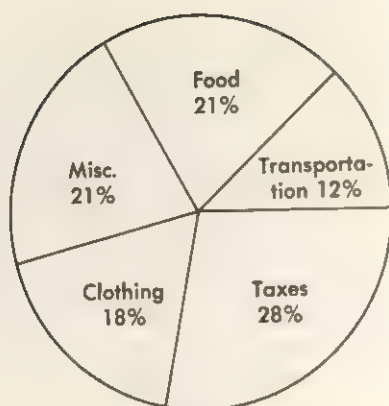
While for many purposes in higher mathematics the tabular display of data is more useful than any other, in the elementary school it may not be as easy for the child to grasp. Only one example is given here—using the height, measured to the nearest inch, of each child in a sixth-grade classroom. These data could be given in tabular form as follows:

HEIGHT IN INCHES	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
NUMBER OF CHILDREN	1	0	0	2	2	1	3	4	5	3	4	0	1	1	0	1	1

These same data may be presented more graphically by the use of a bar graph. The graph does not differ in very many respects from the table, but the child tends to understand better what he can see.

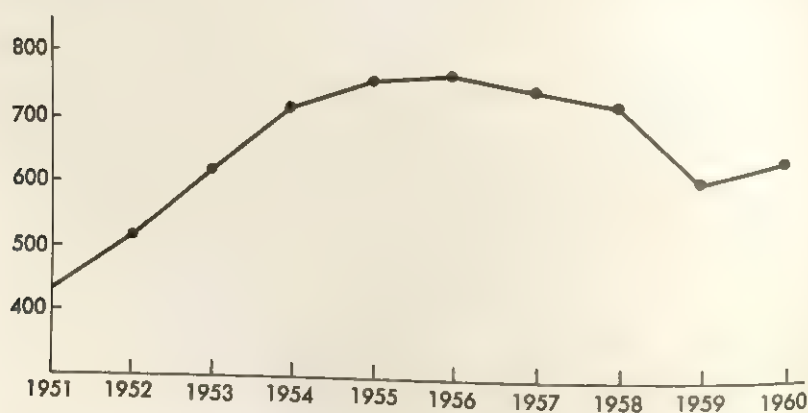


Other kinds of graphs represent other kinds of data more understandably. We are all familiar with the segmented pie diagram or pie graph to represent fractions of dollar expenditures or incomes. The child might find what approximate part of the family income is spent for housing, food, clothing, transportation, taxes, and everything else. His graph might be based on the fraction of each dollar, providing an opportunity to explain that each individual dollar is not so divided, but that the model represents a comparison of expenditures for various items with total family spending. The model might look like this if it were done after the child had learned about percentages.



Many other kinds of graphs can be made, such as the broken-line (or polygonal) graph illustrated below. These graphs are frequently used to indicate business trends, but one might equally well be constructed to represent the trend of gain or loss of scholars of the Shadyside School.

YEAR	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968
NUMBER OF SCHOLARS	432	510	620	719	763	771	752	721	605	637



The construction of graphs and tables should have a two-fold purpose. It should enable the child to organize and interpret (in terms of trends, possibly) the data he has collected, and it should enable him,

once he has had experiences with graphs and tables, to interpret those which he encounters in magazines and newspapers and on TV. He should realize, for example, the importance of broken-line and bar graphs. It is through the distortion of the scale and the failure to indicate scales that graphs can be used to distort the true picture of a situation. Even the graph of the number of scholars in Shadyside School given above can be misleading, since a student might say that the graph indicates by its height that in 1960, there were more than twice as many scholars as in 1959. This emphasizes the importance of the recognition of the zero position on the scale. The data as plotted in Figure 6-1 appear to report relatively stable attendance throughout the week. On the other hand, the same data plotted on the distorted scale in Figure 6-2 appear to report great variations in attendance throughout the week.

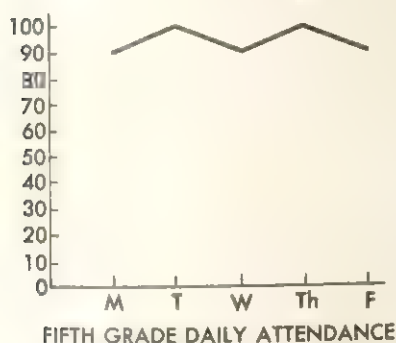


FIGURE 6-1

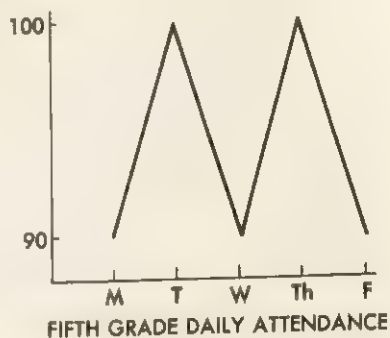


FIGURE 6-2

While the subject of distorted interpretations of data is being discussed, we might return briefly to the second question asked at the beginning of the chapter. Is every seventh person on earth an Indian? It may be true that one-seventh of the earth's population is Indian, but it is far from true that every seventh person is an Indian. This is merely another example of misinterpretation of data.

Under the general heading of organization and interpretation of data, some attention should be given to two related problems. When collecting data is it wise, though perhaps not essential, to collect only pertinent data. However, in a particular problem until a great many data have been collected and analyzed it may not be apparent which data are pertinent. This gives rise to the second problem—the formulation of problems. Again, at the outset, many problems are badly stated

and poorly formulated. Only after some collection and analysis of data can we get a clearer picture of the problem to be faced and solved. All of our decisions in real life are based on uncertainties. We cannot foresee all future consequences of an action. We can formulate hypotheses, validate or invalidate them to the best of our ability on the basis of collected data, and arrive at conclusions which have probable validity. (Here we see that the subject of probability is intimately associated with statistics in decision-making situations.) The child should be given opportunities to collect data for solving simple problems, and in these simple situations he should be given guidance in selecting pertinent data. Only through such experiences can he build a background of confidence, intuition, and practical experience on which to base his future decisions about the pertinence of data.

A final phase of collection and interpretation of data which merits some attention here and in the elementary school classroom is that of finding a way to give a brief but meaningful summary of collected data. One meaningful condensation of many data is the *average* or arithmetic mean. We are all familiar with average heights, average grades, average speeds, etc. They do not give a full picture of the situation being considered, but they do provide a type of summary. For the data on the height of sixth-grade children given earlier in this chapter, we could say that the average height of the children in this classroom is very near  $55\frac{1}{2}$  inches. We should recognize that no single child in the room is necessarily exactly this tall. (Here it would be wise to point out the fallacy of applying averages to individual children or situations.) The average is merely a summary which may be so brief as to be meaningless, but it is a useful datum in itself if properly interpreted. Another such useful datum is the *median* (or middle) height of the group. The median height is 56 inches—halfway between the least and greatest heights. There are some statistical situations in which the median is more meaningful than the mean (or average). The reader is encouraged to read the entire chapter (pp. 272-326) on statistics in the Twenty-fourth Yearbook of the National Council of Teachers of Mathematics, *Growth of Mathematical Ideas, Grades K-12*. The section on measures of central tendency contains several good examples related to the mean and the median (and also the *mode*, which is another measure of central tendency). The computation of the *mean* must be left until after the student has learned how to divide, but the *median* may be obtained earlier.

The first question asked in the chapter has been answered. The question could be rephrased: "Must any single datum equal the mean?"

From a single example we see that the answer is no. An even simpler example is this. The average of 7 and 13 is 10 which is equal to neither. Of course, we hasten to admit that some datum may equal the mean. This is true in the case of the three numbers 7, 10, 13.

## INFERENCE AND DECISION-MAKING

It has already been noted that the collection of data is related to the more complex problem of decision-making. If we note that three persons whom we know have received Salk vaccine shots to give them a measure of immunity from polio have not subsequently been stricken with the disease, can we *conclude* that the Salk vaccine prevents polio? Most certainly not! But if we observe that of the millions who have been immunized, only a few have subsequently contracted the disease, we believe that the shots give immunity. There is a high degree of *probability* that they do.

This preceding paragraph brings up the subject of sampling. How large and how representative a sample is necessary before there is a high degree of probability that some occurrence will or will not happen? In modern statistical theory, great progress has been made in answering this question. The methods are far too sophisticated to be of use in the elementary school, and yet an awareness of the problem and experiences with partial solutions of it can be given the elementary school child. He can be led to the understanding that the observation of three sunny days hardly gives a basis for expecting the fourth day to be sunny. At the same time, he will believe after observing that for several years no more than two days in July have been rainy, there is a great probability that any given July day on which his family plans a picnic will be a sunny day. The child must also be aware of the *representativeness* of the sampling. For example, suppose an interviewer asked only women to state whether they like baseball. He might take a very large sample, but if he found that most of his sample did not care for baseball, he could hardly conclude that the sport is universally disliked. His sample was biased—not representative of the entire population. A miniature copy of this example could be duplicated in the classroom to illustrate the necessity of choosing samples carefully when the entire population cannot be polled.

In the upper elementary grades the child may be given exercises in the formulation of hypotheses from a partial examination of some collected data and in testing the validity of his hypotheses on the basis of the available data. If, from a partial examination of the data given

early in the chapter, he formulates the hypothesis that most of the children in a sixth-grade room are less than five feet tall, he will find that this hypothesis is actually borne out by a full examination of the data. However, if in the collection of data, he had measured the height of the five tallest children first and on the basis of this early data had predicted that most children in the room were over five feet tall, his hypothesis would have been invalidated by the subsequent observation of all data.

We might say that the child must eventually be able to make decisions on the basis of the best information available to him. He will not necessarily have knowledge of the validity of the data available to him. He should have been given experiences at an early age which would condition him to accept advertising claims, for example—frequently based on biased rather than representative data—with a healthy skepticism. He should begin even in the elementary school to evaluate financial data, to seek out information and try to evaluate some elementary political claims, to try to evaluate conflicting reports, about such things as school improvements, by collecting and organizing pertinent data. He should be given an opportunity to develop a realization that data-processing and decision-making are ever broadening fields in which there is much known now and much more to be developed in the future.

## EXERCISES

1. Find the average of these five numbers:

72	87	66	100	91
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2. Find the average of these daily attendance records:

27	27	30	29	26
25	28	29	30	30
27	26	24	23	24

3. The scores on a mathematics test of a class were:

84	83	81	95
63	72	84	79
98	84	75	71
74	65	52	78
78	77	84	81
83	82	77	86
87	86	72	91
94	91	96	

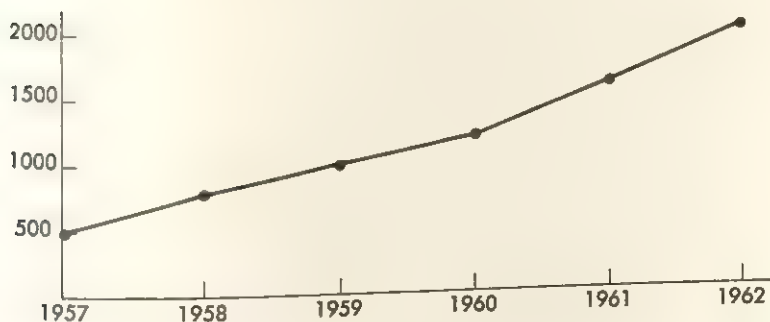
- (a) Rank the scores from highest to lowest.
- (b) Find the range of the scores.
- (c) Find the median of the scores.
- (d) Find the mean of the scores.
- (e) Plot the class scores on a bar graph.

4. Define an average in language suitable for a fourth-grade child.
5. The daily attendance figures in all fifth-grade classes at Hillside School were as follows:

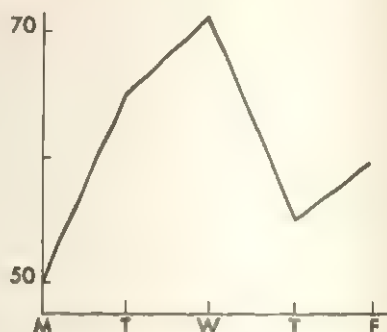
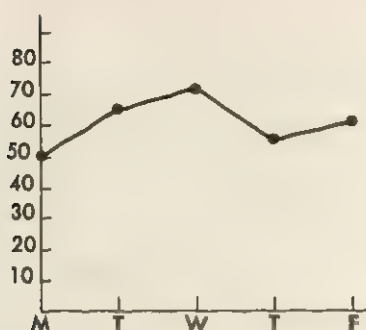
Monday	Tuesday	Wednesday	Thursday	Friday
56	60	63	61	58

Draw a bar graph to represent this record.

6. Draw a broken-line graph to represent this record.
7. The sixth-grade classes at Hillside School studied the home budget. They found that the average family spent  $\frac{1}{4}$  of its income for housing,  $\frac{1}{6}$  for food,  $\frac{1}{10}$  for clothing, and the remainder for all other things combined. Draw a circle graph to represent this data.
8. By interviewing your classmates find the average amount spent daily for one week by members of your class for: (a) lunch, (b) entertainment, (c) tobacco, (d) snacks, (e) transportation, etc. Represent these data graphically. Compare and criticize the graphs.
9. Discuss the trends in enrollment at Jones College shown by the following graph.



10. Discuss the relative merits of the following two graphs showing the same data.



11.



- In what year did the Dow-Jones Industrials reach its peak?
  - How many points did the market rise in the ten year period?
  - What happened to the value of stock during 1960?
12. If you know that the average score on a test taken by 1,000 students is 78, and that a given student made a score of 82, discuss his standing in the group.
13. Collect a set of data. Find the mean and median. Use two types of graphs to present the data graphically.
14. Discuss the relative merits of bar graphs, pictographs, pie graphs, and broken-line graphs for representing various kinds of data.

15. Can the examination of any finite number of cases ever prove a proposition about an infinite set? Explain.
16. Can the examination of any finite number of cases ever disprove a proposition about an infinite set? Explain.
17. Discuss one application of statistics to each of five different areas of science, business, or the humanities.

## Extended Activities

1. Measure the weight of 10 adults. Find their average weight.
2. With reference to the weights in Extended Activity 1, do any of the persons weigh the same as the average? Must some person weigh the average weight?
3. Find the differences between the weights of the persons weighed in Extended Activity 1 and the average weight. Add the negative differences. Add the positive differences. Does the sum of the negative differences exactly cancel the sum of the positive differences? Explain.

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## Chapter 7

Concepts to be developed in this section are:

1. *The musical tones are related to one another mathematically in terms of vibrations per second.*
2. *Artistic perspective is achieved through the use of geometric principles.*

## Understanding the Broader Applications of Mathematics

### I. FINE ARTS

The purpose of this section is to focus attention on a few basic relationships between mathematics and two of the fine arts—music and painting.

#### MATHEMATICS AND MUSIC

In 1581, while watching a chandelier swinging to and fro in the cathedral of Pisa, young Galileo made a significant observation. He noted that regardless of the width of the “swing,” the time consumed in completing each oscillation was the same. Further investigation by other scientists revealed a second phenomenon: the frequency of oscillation within a given time period is dependent on the length of the moving pendulum.

The connection between Galileo’s law of the pendulum and musical sounds is seen when we note that sound is produced by a similar vibratory or oscillatory motion. The frequency of vibration of a string

is uniform because it produces a given tone regardless of the loudness of that tone. Its pitch is dependent in part upon the length of the string. This tone-producing motion, like pendular motion, can be described by a sine curve, as illustrated in Figures 7-2 and 7-3. Figure 7-1 represents a tuning fork. As the fork is struck, vibrations are set up and a musical tone is produced. The tone gradually dies away, but remains at its original pitch as long as the fork is vibrating. One complete vibration or *period* is described by the set of movements,  $a \rightarrow b$ ,  $b \rightarrow c$ ,  $c \rightarrow a'$ . Figures 7-2 and 7-3 represent graphs taken of the fork's vibration at its inception and near its "death." What do we observe? Probably the most noticeable feature of both graphs is their *regularity* or *periodicity*. It is this periodicity of vibrations which distinguishes musical sounds from noise. Although the tone-producing vibrations of most musical instruments are described by curves more complex than the simple harmonic curve of the tuning fork, the vibrations are all characterized by regularity.

Tuning Fork



FIGURE 7-1

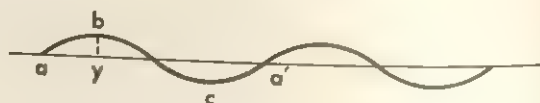
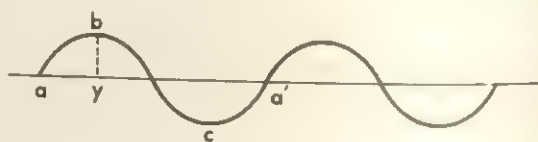


FIGURE 7-3

A comparison of Figures 7-2 and 7-3 with each other and with our description of the perceived musical sounds they represent will also indicate how pitch and amplitude (loudness) are determined. We said that the pitch of the tone represented by Figures 7-2 and 7-3 remained constant, while the amplitude diminished. In examining the graphs, we note that the length of time required to complete each period represented by  $a - a'$  is constant for the two, while the amplitude (represented by  $b - y$ ) diminished from Figure 7-2 to Figure 7-3. If one complete vibration consumes  $\frac{1}{500}$  second, the rate or *frequency* of

vibration periods during a one-second interval would be 500 in both instances. Since both the frequency of vibration, as described by the graph, and the resultant pitch which we hear, remain constant, we see that *pitch is determined by frequency of vibration*. Similarly, since the amplitude diminishes, both between the vibrations of Figures 7-2 and 7-3, and between the sounds thus produced, we see that the *loudness of a tone is related to the amplitude of vibration*. Although loudness and the amplitude of vibration are not directly proportional, we can say that, in general, the harder we toot, whistle, plunk, or boom, the greater the amplitude of the vibrations and of the resultant perceived sounds.

If pitch is determined by frequency of vibration, what determines frequency of vibration? We have already partially answered this question in our reference to early discoveries that frequency of oscillation is dependent on the length of the vibrating body. Most of us have observed the relationship between the size of musical instruments and the tones they produce. A violin, for example, has much shorter strings (vibrating bodies) and thus a much higher pitch than a bass viol. A flute's tones are high, but a piccolo's, whose air column (vibrating body) is shorter, are even higher. A big bass horn, whose air column is so long that it must be wound around and around, produces tones of low pitch, while the much smaller trumpet's range is relatively high.

It would be difficult, if not impossible, to construct all these instruments so that they might be played together harmoniously were it not for the fact that there is a precise mathematical relationship between the length of the vibrating body and its frequency, and thus, its pitch. Stated in mathematical terms, the frequency varies inversely with the length of the vibrating body, other things being equal. That is, doubling the length of a string, air column, etc. halves the frequency; halving the length doubles the frequency, as illustrated in Figure 7-4.

When the frequency is doubled, the pitch is raised by exactly one octave; when the frequency is halved, the pitch is lowered one octave.

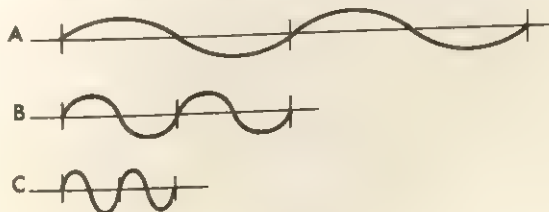


FIGURE 7-4

Thus, if middle C has a frequency of 256 vibrations per second, we know that the tone produced by a frequency of 512 (twice 256) is a tone one octave higher than middle C. This relationship is usually stated as a ratio 2/1. Similarly, the relationship between other pairs of tones (intervals) can be expressed by their vibration fractions, as illustrated in Figure 7-5.

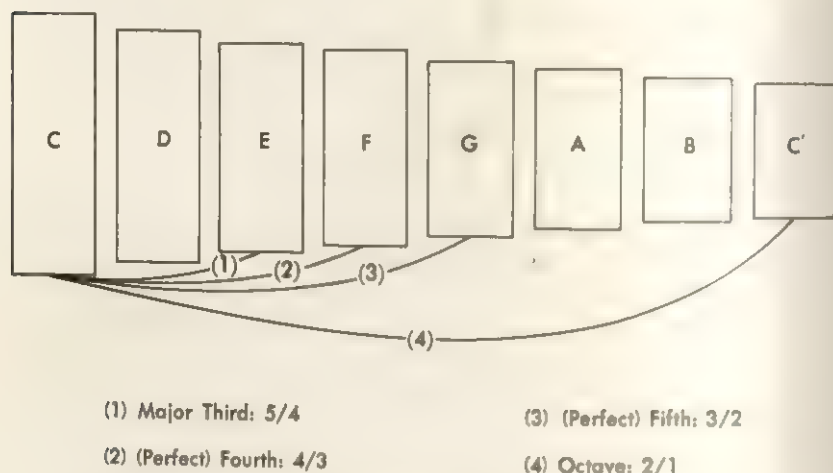


FIGURE 7-5

On the simple one-octave xylophone represented in Figure 7-5, we can see that the lengths of the vibrating bodies (bars) demonstrate concretely these ratios. The C bar which vibrates at 256 per second is twice the length of the C' bar which vibrates at 512 per second. The C bar is  $\frac{5}{4}$  the length of the E bar, etc.

There are many other mathematical relationships significant to the production of musical sound. For example, in the discussion of the relation of a vibrating body's length to its pitch, we qualified the statement of inverse relationship by the phrase "other things being equal." These "other things" include weight, density, and elasticity of the vibrating body, each of which also bears a precise mathematical relationship to pitch. We have all noticed the variation in the dimensions of a bass drum and a snare drum, and between strings which produce tones of low and high pitch. Many of our instruments would look strange and, indeed, some probably would be impossible to construct, carry, or play, if vibration frequency were determined solely by length.

We can only mention another broad area of music closely related to mathematics—*harmonics*. The fact that the vibratory bodies of most

musical instruments vibrate, not only as a whole, but also in parts, means that what we hear as a single tone is usually a complex of pitches also mathematically related to the complex of vibrations. Because these "overtones" or partials vary in their presence and strength with the construction of the instrument, each instrument has a characteristic tone quality. By determining the simple components of a complex tone, the tone may be simulated by a proper combination of pure tones. Thus, there are organs which can simulate the tones of many orchestral instruments. In recent years, a great deal of experimentation based on these relationships has been done in producing "new" sounds which may, in time, add more dimension to our musical pleasure. Harmonics also significantly influence the smoothness (*consonance*) or harshness (*dissonance*) of chords.

Now let us briefly turn our attention to a consideration of rhythm, another important component of music. The very definition of rhythm as "measured movement" implies a quantitative relationship. Whether or not we are familiar with musical notation, most of us have "felt" (psychologists tell us with actual muscular response) various aspects of rhythm in listening to music: *tempo* which refers to the comparative fastness or slowness of the pace of movement; the *pulse* or beat which is the regularly recurring unit of measurement; the recurring *accent* given certain beats which, in turn, helps us group units or beats into matching sets called *measures*. In the over-all structure of musical compositions there is a measured recurrence of sets of measures into phrases, sentences, and musical ideas. We have also noticed that notes of different time value bear specific fractional relationships to one another.

A large part of the vocabulary of music consists of mathematical terms. Time values of notes are expressed as unit fractions, each succeeding smaller unit having one-half the value of the preceding as follows: whole note (♩), half note (♪), quarter note (♫), eighth note (♬), sixteenth note (♭), thirty-second note (♭), etc. Periods of silence (rests) are similarly expressed in the same fractional progression. A dot following a note raises its value by one-half. Thus a dotted quarter note (♫.) has a time value of one quarter note plus one eighth note, or three-fourths of one half note.

The *time signature*, expressed as a fraction with the unit as denominator and the "set" as numerator, gives us the unit of measure (the beat) and the number of beats to a measure. A time signature of  $\frac{2}{4}$ , then, means that one quarter note (♫) constitutes one beat and that there are two such beats to the measure, or  $2 \times \frac{1}{4}$ .

These examples, though by no means exhaustive, should indicate the great extent to which mathematics contributes to music's rhythmic and notational aspects. However, it should be noted that musical notation, which like the written language of mathematics has a long history of development, is not itself music, but merely facilitates communication of musical ideas from creator to performer to listener.

## MATHEMATICS AND PAINTING

The intellectual life of the Middle Ages was devoted almost exclusively to the doctrines of the Christian church. The work of artists of the period reflected that devotion. Prominence of individuals pictured in the paintings of this period and in earlier Greek and Roman art was often determined by the religious or sociopolitical importance of the individual rather than by a realistic portrayal of the scene. With the coming of the Renaissance, artists made great progress in the representation of realism in painting. Kline in his *Mathematics in Western Culture* attributes this development during the Renaissance to certain characteristics of the period: the renewal of interest in the physical world and a desire to portray it realistically; revival of reverence for Greek philosophy, including the belief that the true significance of reality was revealed by stripping it to its bare mathematical structure; and the fact that the outstanding artists of the period were also its chief architects, engineers, and mathematicians.<sup>1</sup>

The realism captured in the paintings of the Renaissance masters is due largely to the discovery and use of perspective. (In Figure 7-6 the flat characteristic of non-perspective painting is well illustrated.) Defined as the art of making things appear to be three-dimensional when presented on a two-dimensional canvas, perspective was studied assiduously by many of the great painters of the period. Leona Battista Alberti in a treatise on painting written in 1435 said that the first requirement of the painter is to know geometry. Piero della Francesca was a great painter who mastered the art of perspective and was one of the best mathematicians of the fifteenth century. Leonardo da Vinci in his *Trattato della Pittura* began with the words, "Let no one who is not a mathematician read my works."

These geometric rules of perspective are illustrated in Figure 7-7 which shows both one-point and two-point perspective. It can be seen

<sup>1</sup> Morris Kline, *Mathematics in Western Culture* (New York: Oxford University Press, 1953), p. 126.

that in such a diagram parallel lines converge at a vanishing point on the horizon. This represents reality in that objects appear to become smaller as they recede farther and farther into the distance. Thus artists achieve a sense of perspective through the representation of



Courtesy of National Gallery of Art, Washington, D.C., Samuel H. Kress Collection

FIGURE 7-6. *"Madonna and Child Enthroned," Margaritone.*

parallel lines by lines which, if continued, would converge at a vanishing point, or through diminishing the size of objects as they appear to be more and more distant from the observer. Examples of perspective achieved through each of these techniques are shown in Figures 7-8

and 7-9. Artists also achieve a sense of perspective by making objects less distinct in both line and color as they recede into the distance, but this technique is of less interest to us here.

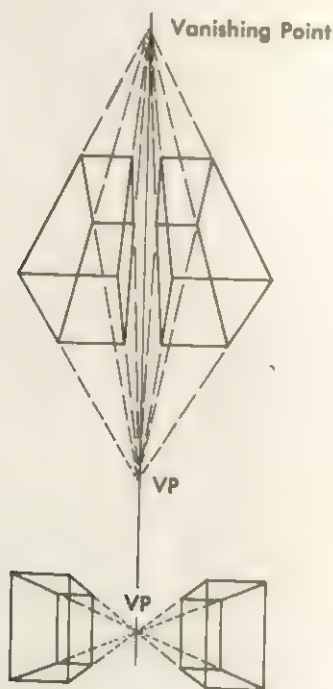


FIGURE 7-7

The German artist, Albrecht Durer, was among the first to conduct extensive experiments in perspective. He conceived the canvas as a glass screen through which the scene to be painted was viewed. (This technique is illustrated in Figure 7-10.) Durer actually used a grid on a glass pane through which he viewed his subjects. Holding one eye fixed, he imagined that lines of light (a projection) radiated from that eye to all points of the scene. Where each line struck the glass, he placed a point. This set of points, or *section*, was then transferred to canvas, producing an impression on the eye meant to be identical to that created by the scene.<sup>2</sup>

Durer's work had both importance and limitations. Eventually, by the use of mathematics, his elaborate apparatus was eliminated, and

<sup>2</sup> Kline, *Mathematics in Western Culture*, p. 135.

principles applicable to the depiction of any scene, real or imaginary, were established.



Courtesy of National Gallery of Art, Washington, D.C., Samuel H. Kress Collection.

FIGURE 7-8. "The Annunciation," Master of the Barberini Panels.

## SUMMARY

Through the ages, the expression of creative ideas through music and painting has been implemented by mathematics.

The science of acoustics, developed through the application of mathematics to physical laws, has contributed much to music. The construction of a broadening range of standardized musical instruments, differing in pitch and tone quality, is made possible through the application of acoustical principles. Knowledge of these principles is used by the composer in structuring harmonies which will convey the mood or feeling he wishes to express, and by the performer in playing and tuning his instrument. Through the employment of fractional concepts in musical notation, mathematics enables the composer to communicate the rhythmic aspects of a musical idea.



Courtesy of National Gallery of Art, Washington, D.C., Samuel H. Kress Collection

FIGURE 7-9. "The Visitation with Two Saints," Piero di Cosimo.



FIGURE 7-10.

The history of painting reveals that the application of geometric principles to painting during and following the Renaissance has significantly increased the realism with which an artist may represent his idea. The technique of perspective, the creation of a three-dimensional illusion on a plane surface, was pioneered and perfected by such Renaissance artist-mathematicians as Leona Battista Alberti, Piero della Francesca, and Leonardo da Vinci.

Mathematics in the hands of a musician or painter is a tool which broadens his range of expression, enabling him to convey more precisely his artistic ideas; and through the application of mathematics to art, a broader, richer field of aesthetic enjoyment is provided for everyone.

## II. ASTRONOMY

Concepts to be developed in this section are:

1. *The laws of astronomy are expressed mathematically.*
2. *Mathematics has enabled astronomers to make discoveries and predictions.*
3. *In the space age, the astronaut's survival as well as the layman's ability to orient himself to his expanding universe will depend greatly upon knowledge of mathematics, astronomy, and related sciences.*

The reader (and the child in the classroom) may wonder what a professional astronomer actually does. What does he need to know? How much training in mathematics does he need? How does he use his knowledge of mathematics in his work?

Whether his field is descriptive astronomy, celestial mechanics (mathematical astronomy), astrophysics (physical astronomy), or another, the professional astronomer is concerned with development, research, and discovery. He uses known principles to point the way toward new discoveries; he searches for new hypotheses which may be supported by experimental data and which will explain previously unexplained phenomena; he uses both mathematics and astronomical principles and devices in seeking new data or phenomena.

An example of these activities is found in the means by which two of the planets of our solar system were discovered. Until almost the middle of the nineteenth century, Uranus was thought to be the outermost planet of our system. It was noted by astronomers, however, that its actual orbit deviated from its expected course as if the planet were

being attracted by some other body. On the basis of this deviation, two young mathematicians, Adams of England and Leverrier of France, made independent mathematical predictions of the location of the attracting body. The German astronomer, Galle, found the attracting body, the planet Neptune, with his telescope in almost the exact location represented on paper by the calculations of the mathematicians. The planet Pluto was discovered as recently as 1930, in much the same manner. Similarly, in 1801, the asteroid Ceres was found between the orbits of Mars and Jupiter because the presence of a planet in this position was predicted mathematically. Since that time, more than 1,000 asteroids have been found in this region. Some astronomers feel that they may be remains of a planet which exploded millions of years ago.

The various methods by which distances to other bodies are calculated provide other illustrations of the application of mathematical (and physical) principles to problems of astronomy. The distance to the Moon, for instance, may be calculated by the *parallax* method, based on the fact that an object, such as the Moon, when viewed from different positions, seems to shift locations relative to other background objects. (By holding up a finger a few inches from your face and viewing it, first with one eye and then with the other, you can demonstrate this "shift" for yourself. You may also note that the farther away you hold your finger, the slighter the shift becomes.) The Moon's location among the stars as viewed from two positions on Earth 4,000 miles apart differs by almost 1 degree, or  $\frac{1}{180}$  of the arc of the sky seen from any point on earth. This small angle of apparent shift, or parallax, forms the apex of a triangle of which 4,000 miles (the distance between the two observation points) is the base. Given the base and the angle of the apex of the triangle, the length of the sides, or distance to the Moon, may be determined mathematically.

Because of the relatively greater distance to the Sun and the smallness of Earth, the apparent change in the Sun's position viewed from two points on Earth is too slight to be useful in determining its distance. The planetoid Eros, however, which orbits between Earth and Mars, has been used as a stepping stone in determining the Sun's distance by parallax. By finding the distance to Eros at its nearest position to Earth, then observing its motion around the Sun, Eros' distance from Earth may be converted to a fraction of the distance between Earth and the Sun. This fraction is used to compute the total distance to the Sun from Earth.

The parallax method has also been used to find the distance to some

of the nearest stars by utilizing the diameter of the Earth's orbit, 186,000,000 miles, as the base.

In determining the vast distances to the more remote stars, another method, also involving mathematics, must be employed. Most of us have observed that a distant light appears less bright than a close light of the same brightness. Using mathematics, the physicist can compare the apparent brightness of the two lights to determine their distances. In much the same way, variable stars which have been observed in many galaxies, including our own, are used to determine distances. Variable stars change periodically in brightness. The intervals between bright stages, or periods, of these stars are the same for stars of the same brightness; thus, a star's absolute brightness may be determined by observing the length of its period. A star's *absolute* brightness and *apparent* brightness are then compared to determine its distance.

Many other instances of the important role of mathematics in astronomy could be cited. The speed at which a star is moving toward or away from Earth, for example, is calculated mathematically from observations of its light spectrum. In speeding toward us, a star (or any other light source) pushes into its own light waves, compressing them and causing the entire light spectrum to shift slightly toward the shorter or blue waves. In moving away, the light waves become elongated and shift toward the red end of the spectrum. Since the shift is proportional to speed, the motion toward or away from us and the speed of a star in miles per second can be computed from measurements of spectrum shifts. (Incidentally, all stars in the other galaxies show red shifts, indicating that they are all moving away from us, some as fast as one-fifth the speed of light.)

Mathematics, physics, and electronics have led astronomers to many discoveries in the field of radio astronomy—among them, the detection of strong radio signals from areas where no stars are visible. Astronomers are considering as possible sources of these signals old stars which no longer send out light; new stars whose light has not yet reached Earth; and clouds of cool gas, chiefly hydrogen, which emit no light waves.

Recently, many men and women with training in mathematics, astronomy, and physics have been engaged in the computation of the orbits of artificial satellites, in the design of experiments to be performed by satellites, and in the correlation of data gathered and transmitted to Earth from satellites. New understandings about our solar system have been gained through these efforts.

The mathematical training of the modern professional astronomer must include the theory of relativity, non-Euclidean geometry, and

celestial mechanics. These are based on a sound foundation in the axiomatic treatment of geometry, calculus, analytic geometry, and trigonometry. These, in turn, are founded on a comprehensive knowledge of algebra and arithmetic. As has been noted in Chapter 3, algebra is merely a generalization of arithmetic. It appears, then, that all of the astronomer's knowledge of mathematics is based on a strong and enduring foundation of arithmetic—the chief mathematical subject of the elementary school.

## THE ASTRONOMY OF THE ELEMENTARY SCHOOL CHILD

What astronomy is within the experience and ability of the elementary school child? What mathematics does he need to know to comprehend, investigate, and communicate facts about astronomy? The following list includes some of the facts and topics appropriate for the elementary school child.

1. Distance of planets of our solar system from the Sun and from each other (see Figure 7-11)
2. Relative sizes of planets
3. Distance to and size of Earth's moon
4. Orbits of planets in our solar system
5. Orbits of satellites
6. Courses of comets
7. Solar and lunar eclipses
8. Place of our solar system in the Milky Way Galaxy
9. Relative distance to and sizes of stars
10. Meaning of light-year

It seems hardly necessary to point out that many mathematical concepts are related to this brief list. One of the first which comes to mind is the ability to read, write, and interpret numerals which denote large numbers. This ability might well be developed by using the distances referred to in the above list. A chart, such as the one in Figure 7-11, showing the distances from our sun to the planets of our solar system, might be constructed as an integrated science-mathematics project. (It should be noted that the chart in Figure 7-11 is not drawn to scale since the mean distance from the Sun to Mercury is only about  $\frac{1}{100}$  of that from the Sun to Pluto.)

Table I contains some additional information which might be useful in a mathematics project involving the planets of our solar system.

Although it is doubtful that the human mind can comprehend such distances as 3,670,000,000 miles (relatively small in astronomy) in the same sense that 2 feet can be understood, the use of reference measures and models can be of considerable value. References such as the following may help children grasp relative sizes and distances.

1. More than a million Earths could be placed inside the Sun if it were hollow.
2. If Earth were the size of a quarter, the relative size of the Sun would be a nine-foot ball about three football fields' length (1,000 feet) away. Pluto would be about seven miles away!

By expressing various relative sizes and distances included in Table 1 as ratios, children may devise their own references. For example, using the diameter of an orange, basketball, beach ball, or other familiar sphere to represent that of the Sun, they could approximately determine the relative size of Earth in terms of pinheads, thumb's widths, pencil points, etc. Comparisons of the relative size of Earth and its moon might be made. The distance between Earth and its moon and that between Jupiter and its moons could also be compared. To make comparisons of distance within our solar system more easily understood, the *astronomical unit*, or distance from Earth to the Sun, is useful. The distance from Pluto to the Sun, for example, may be expressed as approximately 40 astronomical units.

A larger unit necessary for efficient representation of greater astronomical distances is the *light-year*. Light travels at approximately 186,000 miles per second, and a light-year is the distance light can travel in one year. The use of light-year as a unit of measure is comparable to the establishment of a larger unit in a multiplicative system of numeration. The nearest star (except the Sun) is approximately

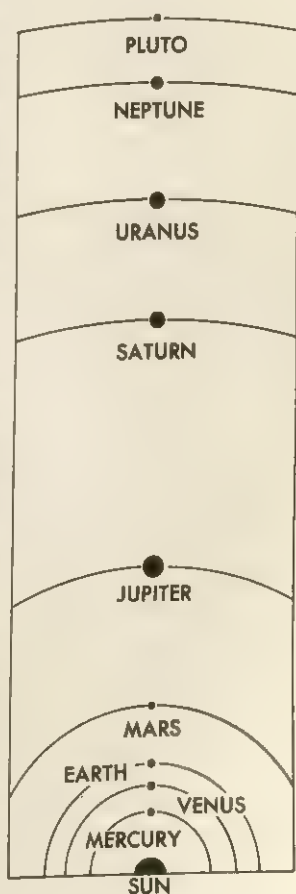


FIGURE 7-11

PLANET	MILLIONS OF MILES FROM SUN	DIAMETER IN THOU- SANDS OF MILES*
Mercury	36	3.1
Venus	67	7.7
Earth	93	7.9
Mars	142	4.2
Jupiter	484	88.7
Saturn	887	71.6
Uranus	1787	32.0
Neptune	2797	31.0
Pluto	3675	3.6

\* The diameter of the Sun is about 865,000 miles.

TABLE 1

4.1 light-years away. (Expressed in relative terms, if our solar system were the size of a quarter, this star would be almost the length of a football field away. One of the "nearer" galaxies to our Milky Way is 150 million light-years away, while some of the farthest known are several hundred million light-years away.

Other mathematical concepts, particularly those related to measurement, contribute to understandings of astronomy, and vice versa. That our calendars are based on the Sun's position and our clocks set by the passages of bodies over fixed points illustrate man's use of astronomy in developing units of time measure. The fact that the relative positions of many stars appear "fixed" as represented by the Greeks, whereas they are actually traveling at tremendous speeds, gives some concept of the vastness of the universe and contributes to the child's understanding of the approximate nature of measurement.

In order to comprehend, investigate, and communicate adequately in the field of astronomy, the child needs skill in computation and in measurement of angles. If the daily paper tells him that at 7:20 P.M. local time a given man-made satellite will be visible 79 degrees above the

horizon as he faces south, he should be able to interpret this information with sufficient intelligence to enable him to see the satellite (all other factors being favorable).

The elementary school child also should be made aware that the courses of comets, the orbits of moons and planets, and the motion of heavenly bodies in general are governed by astrophysical laws which may be stated in mathematical form. It is because of this regularity of behavior, which may be expressed mathematically, that professional astronomers are able to predict such occurrences as eclipses, appearances of comets, and times when certain other astronomical phenomena may be observed. Such an awareness should enable the child to approach the whole subject of astronomy (and space travel) with a more realistic attitude toward the work of the astronomer and the astronaut.

## SPACE EXPLORATION

The day is rapidly approaching when man can and will travel to the Moon and then, possibly, to other planets. What mathematical equipment will he need to embark on such a momentous journey?

If he is to survive his space explorations and not become a bit of lifeless cosmic matter, the astronaut of tomorrow will need comprehensive training in astrophysics. Logically this training relies on his mathematical background which begins at the elementary school level. It is becoming increasingly important to provide the best possible teaching and direction as foundations for more advanced study.

High speed computers will be essential for the rapid assimilation of information on which decisions will be based. Many problems of guidance and true celestial navigation will be handled entirely automatically through the use of computers; but the final decisions which will have to be made before and while man is journeying into space will have to be made by man himself.

## III. PHYSICS AND CHEMISTRY

Concepts to be developed in this section are:

1. *Many of the laws of physics and chemistry are statements about number.*
2. *A knowledge of mathematics is an essential requirement for a knowledge of these branches of modern science.*

Since the beginning of human history, man has searched for the underlying structure on which his world is built. He has sought order

and rationality in the physical world surrounding him. The search for answers to the mysteries of the universe culminates in his statements of principles or formulations of laws which, to the best of his ability to observe and understand, symbolize the interactions among the forces in the world around him. However, he is continually re-evaluating these principles in the light of new observations.

The subject matter of some branches of science is more easily ordered—more easily fit into definite patterns—than that of others. Chemistry and physics are two such comparatively well-ordered branches of science. While we are far from being able to state laws which govern all physical and chemical behavior, we are able to do so better than in those sciences which deal with living and thinking beings.

## PHYSICS

In man's experience with the physical world, he passed through phases in which he observed phenomena and was only able to record the occurrences in his memory, in legend, or in some other manner. He did not understand the phenomena, nor was he able to perceive a pattern. Doubtless when man first heard thunder and saw lightning he did not associate the two, but merely remembered each. Later, he may have grasped the fact that lightning preceded thunder, but still may not have seen causality. That is, his observations were merely quantitative and qualitative, and did not include the cause and effect relationship between the phenomena. However, eventually he was able to grasp that the one preceded the other in a cause-effect relationship.

Each child's observations of the physical world follow much the same pattern. He finds it easier to describe the quantitative aspects of his experience than to describe the qualitative. He finds that the use of number helps him record, remember, and communicate his experiences. The qualitative aspects of observed phenomena are not as easily described numerically and, hence, are harder to understand and communicate.

In man's observations of the world around him he was able to recognize recurrent phenomena. It was these recurrences which led him to search for generalizations. When he finally developed a tentative law (or hypothesis) which seemed to fit his observed facts, he frequently found the symbolism and concepts of mathematics were ideally suited for formulating his tentative law in a comprehensible form.

Having formulated a tentative hypothesis, he searched for new evidence of its validity, or evidence that the hypothesis did not properly describe his physical world. Frequently, these subsequent observations caused him to alter the statement of his law; mathematical symbolism also facilitated these alterations.

In his descriptions of the physical world, man also found that mathematics and its symbolism helped him to make easily communicable and useful definitions. For example, power is defined as the time rate at which work is done. This statement is not as easily comprehended, probably, as its equivalent, that  $P = \frac{W}{t}$ . Of course, it is also necessary that "work" be understood and that the units of  $W$  and  $t$  be given or understood; in this case, too, the definitions may best be made in terms of mathematical symbolism.

If these preceding paragraphs seem somewhat bewildering because of their generality, let us consider an example of an experiment in physics which the elementary school child can perform—one which will enable him to understand more clearly the world in which he lives. This experiment would fit nicely into a unit on observational statistics. The child collects a small group of clean tin cans of various sizes which do not have sharp edges. At his home he partially fills each of the cans with water. He measures the distance from the top of each can down to the water level and records these measurements. He puts the cans in the freezer unit of the family refrigerator overnight. In the morning, he measures and records the distance from the top of each can to the surface of the ice. From these data he may be able to draw some conclusions. It should be apparent at the outset that expansion of the water took place in the freezing process. From the data collected for cans of various sizes, the child may be able to decide whether the expansion was purely a linear expansion or whether it was a volume expansion. He may conclude that the expansion caused the new volume to be approximately  $\frac{12}{11}$  of the original volume. He may see why ice floats in water. (There is no more water in the can than there was previously, but when the water freezes, it occupies more volume; consequently, ice must weigh less per unit volume than water, and hence it must float.)

Many other science-oriented experiences can be planned which display not only physical principles but also the way in which mathematics is involved in recording, explaining, and communicating information about our experiences. The experiences could be centered around simple machines, wheels, inclined planes, pulleys, etc. Problems of speed (linear velocity) could be associated with the number

of turns of a wheel per unit of time (angular velocity). Problems involving work (in its physical sense) as force times distance could be used to show how pulleys and rope fashioned into block and tackle enable us to move heavy objects by decreasing the force and increasing the distance (length of rope) through which the force is applied. The inclined plane also enables work to be done in a similar fashion—increasing the distance while decreasing the force.

After having shown how mathematics is intimately connected with the statement and use of physical principles, the teacher should not find it difficult to discuss the next step: that physical discoveries have been made because mathematics predicted them. In the fields of atomic and nuclear physics this is particularly true. Since atoms, electrons, and nuclei cannot be seen, physicists have had to rely on mathematical techniques to explain and predict phenomena involving atoms and sub-atomic particles.

It would not be incorrect to say that all great physicists have had a broad grasp of mathematics. Probably the greatest physicist of modern times, Albert Einstein, is thought of by most laymen as a mathematician. However, the truth is that when Einstein needed a tool early in this century for expressing his new theory of relativity, he reached back some twenty years into mathematical history for tensor analysis—a branch of mathematics invented by two Italians, Tullio T. Levi-Civita and Gregorio Ricci. With this help he expressed a theory which has worked profound changes on modern day physics.

Mathematics has played an important part in the development of physics. It has been the tool by which the physicist has expressed his generalizations and has been able (to a certain extent) to make a comprehensible and comprehensive structure of his knowledge of the physical world.

## CHEMISTRY

The laws of chemistry are in actuality laws of physics because the behavior of elements is essentially the behavior of physical entities. Still, chemistry is considered by many to be a separate discipline, and to a certain extent it will be so considered here.

Many of the relations, formulas, and principles of chemistry involve some mathematical symbolism in their statements. Some of the types of chemical problems which involve mathematics include:

1. Determining density and specific gravity
2. Writing formulas for compounds

3. Making experimental measurements
4. Writing chemical equations (including the use of valences)
5. Determining weight relations in chemical formulas and in chemical reactions
6. Expressing concentrations of solutions
7. Determining atomic weights.

Let us consider, as an example, some of the elementary mathematics used in writing a chemical equation. If we heat copper with sulfur, cuprous sulfide is formed. We observe this experimentally. We could state the result as copper + sulfur = cuprous sulfide. Instead of writing the words, though, we might write it with chemical symbols. It could be written incorrectly as  $\text{Cu} + \text{S} = \text{CuS}$ . This chemical equation "balances," but it does not agree with the observed facts that were to be expressed by the equation.  $\text{CuS}$  is not cuprous sulfide. It is necessary to "balance" the equation so that the correct number of atoms of copper are combined with the correct number of sulfur atoms to form the observed cuprous sulfide. The correct (balanced) equation is  $2\text{Cu} + \text{S} = \text{Cu}_2\text{S}$ . Similarly, when calcium oxide and hydrochloric acid are combined, they yield calcium chloride and water. The "unbalanced" equation, which merely states the ingredients and the results, might be written as:  $\text{CaO} + \text{HCl} = \text{CaCl}_2 + \text{H}_2\text{O}$ . The chemist (or chemistry student) must select suitable multipliers to cause the equation to be "balanced"—to have the same number of atoms of each element entering the reaction as there are left after the reaction. The balanced equation is:  $\text{CaO} + 2\text{HCl} = \text{CaCl}_2 + \text{H}_2\text{O}$ .

An even simpler use of mathematical symbols in chemistry is illustrated in the old jingle:

Once there was a chemist.

Alas, he is no more.

For what he thought was  $\text{H}_2\text{O}$

Was  $\text{H}_2\text{SO}_4$ .

As we mentioned earlier, the areas of chemistry and physics intersect. Let us consider a problem from physical chemistry which occurs in many advanced mathematics books. It should be understood at the outset that the solution to this problem is well beyond the grasp of elementary school children (and many of their teachers, too), but it does give an indication of the kind of problem in chemistry to which the techniques of higher mathematics are applicable.

Problem: it has been determined experimentally that radium (more properly radium D or radiolead) decomposes at a rate which is proportional to the amount present. (While radium does not decompose or

disintegrate continuously, but instead emits particles discretely, it will be assumed that the amount of radium is so large compared with the size of its sub-atomic particles, that for all practical purposes continuous decomposition results.) If of 100 mg. of radium set aside now, 95.9 mg. remain after one century, how much will remain after  $t$  centuries? How long will it take for half of the original amount to decompose (this is called the half-life, and the concept of half-life is important in atomic physics or chemistry)?

The solution of this problem involves the use of calculus. It is sufficient for our purposes to indicate that if  $q$  represents the number of milligrams of radium present after  $t$  centuries (and  $t$  represents the number of centuries since the original 100 mg. were set aside), then the relationship between  $q$  and  $t$  is given in the following equation:  $q = 100(.959)^t$ . The half-life of radium would be obtained by substituting 50 (milligrams) for  $q$  and solving for  $t$ . The solution is  $t = 16.5$  (centuries).

Mathematics has also contributed to research in chemistry. In 1860 the chemist Dmitri I. Mendeleev, in attempting to arrange the then known elements into some pattern, discovered that if he arranged the elements by their atomic weights, every eighth element (among the first several) had similar chemical properties. He found that a continuation of the arrangement which caused elements with similar chemical properties to appear in every eighth position resulted in some blank spaces. He predicted the nature of the elements which would fill these spaces. Although time and increased knowledge have caused modification of his periodic table, the fact remains that he did use mathematics in predicting the existence of elements many years before their existence was verified and the elements isolated.

Finally, it should be noted that chemistry and physics are most closely allied in the area of molecular physics. Building molecules into the multitudes of synthetic materials which we know today is an accomplishment of men and women who understand the mathematical laws governing the interaction of the physical and chemical factors which are involved in the production of these materials.

## IV. THE OTHER SCIENCES

Concepts to be developed in this section are:

1. *In recent years the need for quantitative analysis has become increasingly acute in the biological sciences, the earth sciences, and the social sciences.*

2. *Methods are being devised by which problems in these fields may be formulated mathematically.*
3. *Mathematical formulation of problems in these fields renders solutions possible by known methods and makes communication more meaningful.*

The cause-effect relationships treated by the sciences considered in this chapter involve growth, change, mutation, thought, and free will; therefore, precise mathematical formulation has been slower to develop in these areas than in the physical sciences. However, in fields such as geology, the mathematical methods of the physical sciences may often be employed. In other fields, statistical methods making full use of the theory of probability are enabling experts to make increasingly accurate statements regarding the probable behavior of large groups of objects or subjects. Let us turn our attention to individual sciences and the application of mathematical techniques to some of their problems.

## THE BIOLOGICAL SCIENCES

How is mathematics used in botany, physiology, zoology, bacteriology, immunology, biochemistry, biophysics, and other biological sciences? How does its use contribute to new developments in these fields? Perhaps an example from the field of immunology may help us answer these questions. A child, one of identical twins, is badly burned and desperately in need of skin grafts. The doctor refuses offers of skin from the parents and requests instead tissue from the patient's identical twin. Why? His decision is based on knowledge of statistical research in the field of immunology. He knows that the probability of finding two persons, other than identical twins, whose bodies will accept skin from one another is perhaps as low as 1 in 300, while chances of success for such an operation between identical twins is much greater.

The statistical treatment of data, then, may serve as a basis for predicting the probability of certain occurrences which, in turn, may make intelligent decisions possible. In many instances, the same data spur further research in the area. Thus, through dissatisfaction with the general low incidence of success in skin grafts between persons, causes and possible solutions are being sought. It has been found that the injection of cells from the kidney or spleen of a dark-furred mouse into a white mouse embryo produces, for the white mouse, tolerance to the tissue of the dark-furred donor. Perhaps further research will make this finding more generally applicable.

One of the most challenging phenomena of the biological sciences, and one which necessitates the continual compilation of new statistical information, as well as new developmental research, is the occurrence

of mutations. These changes in biological structure are inherited from one generation to the next. For example, in determining the effectiveness of an insecticide on a species of insect, a certain death rate may have been recorded for each of 5,000 tests. Some of the insects, however, may have developed a tolerance to the poison and passed this characteristic on to their offspring, so that on the 5,001st test the previous death rate no longer applies. In recent years, a number of infant deaths have been attributed to a strain of staphylococcus bacteria which has developed a tolerance to previously effective hospital antiseptics. Clearly, more research in mutations lies ahead.

Some mutations observed by biologists have, conversely, been turned to man's use. By applying mathematically stated genetic laws, new and superior strains of foodstuffs may be developed. Polled (hornless) cattle and seedless oranges exemplify such advances. Currently, radioisotopes are being used in experiments at national laboratories to produce mutations in plants. This artificial speeding up of mutations makes the development of new plant breeds possible much more quickly and more economically than before.

## THE EARTH SCIENCES

Mathematical methods and techniques help meteorologists, geographers, geologists and other earth scientists to learn, communicate, and predict information concerning our planet.

In predicting weather, for example, at the United States Weather Bureau's approximately 300 weather stations and 12,000 cooperating substations meteorologists gather data concerning temperature, humidity, air pressure, wind velocity and direction from instruments employing mathematics in their conception, construction, and interpretation. On the basis of statistical compilations of these data, the questions of whether to plant, to go flying or boating, to carry a raincoat, to evacuate an area are being answered with increasing accuracy. These predictions are based on physical laws of motion and heat as applied to sets of weather data. Through the work of meteorologists led by mathematician John von Neumann, an outstanding pioneer in the design of high-speed computers, calculations from weather observations may now be made quickly and efficiently by electronic computers. Weather satellites, built and sent into orbit with the help of mathematics, now yield much previously unattainable information concerning how weather is made. Even with these advanced methods, the absence of sufficient data or insufficient techniques of interpretation

sometimes cause the weather forecast to be in error. Research in this area continues, so that someday we may be able to predict weather accurately. When this day comes, credit will be due in large part to mathematics.

In the branch of geography concerned with earth measurement mathematics is an indispensable tool, which is used in obtaining surveys of the earth's surface by triangulation. Finding and delineating the contour, dimensions, positions, etc. of any portion of the earth's surface by this method is based on the mathematical principle that after establishing two sides and one angle or one side and two angles of a triangle the remaining elements may be computed by trigonometry. A network of such triangles comprises a survey. Other surveys, also employing mathematics, are made through the use of aerial photography.

Once geodetic data are obtained and compiled, the task of communicating resultant information, usually by means of scale drawings, rests largely with the map maker. The chief problem of depicting various features of the earth's surface stems from the fact that a sphere such as the earth cannot be represented on the plane surface of a map without distortion of distance, area, or direction. By applying the principles of mathematical perspective and projective geometry, however, a number of projections adequate for diverse needs have been developed. The familiar Mercator projection, for example, though inaccurate in representation of area and distance, shows direction correctly and is thus useful in navigation. On all types of maps, the use of lines of longitude (meridians) and latitude (parallels) as coordinates by which to fix the position of points on the earth's surface is of great value to the reader. Ptolemy, the Greco-Egyptian geographer-geometer-astronomer of the second century A.D., first used these imaginary lines.

One of the unsolved problems of map making is one which is called the four-color problem. Can a map of contiguous regions of any shape whatever (excluding regions which cross themselves) be drawn and colored with no more than four colors so that no two contiguous regions are colored the same color? To date, no map has been imagined which could not be so colored, yet no mathematical proof has been developed which answers the question either affirmatively or negatively.

Experts in the field of geology use mathematics in exploring the history and composition of the earth. Numbers are needed to help define the geological time scale and make it meaningful. (Figure 7-12).

An example of the use of mathematics in solving geological problems is its use in locating earthquakes. Scattered around the earth are seismological laboratories in which measurements of the earth's motions

PERIOD	ANIMAL LIFE	PERIOD BEGAN APPROXIMATELY
Cenozoic	Man Mammals	1,000,000 to 60,000,000 years ago
Mesozoic	Birds Reptiles	180,000,000 years ago
Paleozoic	Insects Amphibians Fish	550,000,000 years ago
Proterozoic	Invertebrates	1,200,000,000 years ago
Archeozoic	None	At least 3,000,000,000 years ago

FIGURE 7-12

are continually made. The chief instrument of measure is the seismograph, which changes ground motion to electrical current, measures the strength of the current, and records the results photographically or by stylus on paper attached to a revolving drum. When an earthquake occurs, the violent shock waves are recorded as zigzag lines, wider in arc than those recorded for small continual earth motions. These shock waves are of two types: primary, which travel through the earth; and transverse, which travel along the earth's surface. Because the time elapsing between the recording of the primary and the beginning of the transverse waves is proportional to the distances traveled, the distance between the seismograph and the seat of shock (the epicenter) can be calculated mathematically. When the radius of the earthquake from each station is established, arcs from the various reporting stations are plotted on maps. The intersection of at least three arcs pinpoints the location of the disturbance (Figure 7-13).

Geologists at work on determining the age of the earth are using the decay rate of radioactive elements found in the earth's crust as a basis for estimates. All rocks and soil, including the ocean floor, contain small quantities of radioactive elements. The atoms of these radioactive elements constantly disintegrate or decay by giving off particles. Each radioactive isotope (variation of an element) has a characteristic

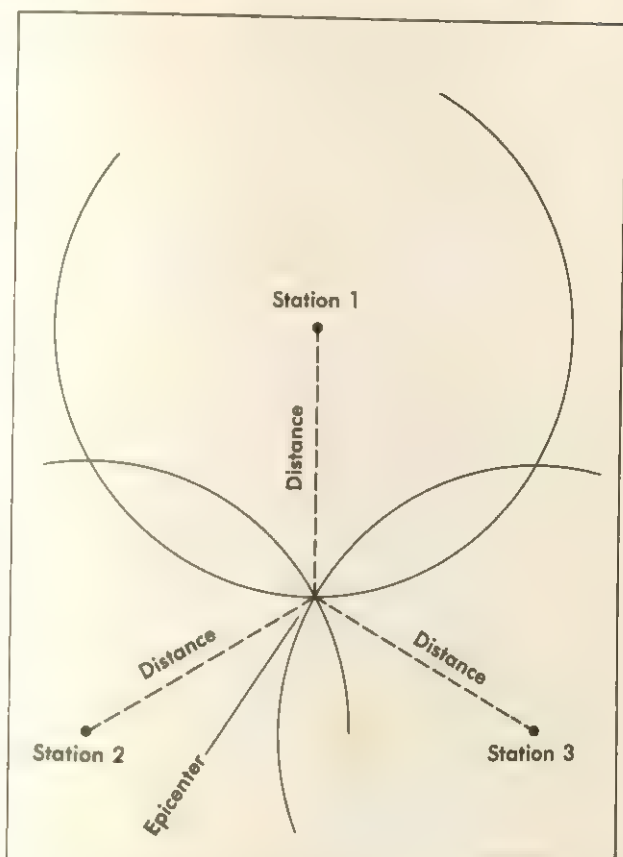


FIGURE 7-13

rate of decay. The unchanging length of time necessary for the isotope to lose one-half its radioactivity is called its half-life. Uranium-238, for example, has a half-life of 4,500,000,000 years; Polonium-214, a half-life of only  $\frac{1}{10000}$  of a second.

Research has determined that an isotope of lead, Lead-206, is formed by the decay of uranium. By measuring the amount of Lead-206 in minerals and also the ratio of this isotope to the uranium present, geologists can estimate the earth's age. Sand from the ocean floor, composed of rock fragments from all over the globe, is the earth sample most frequently used. Meteorites, which scientists believe are representative of the earth's mineral composition at formation, are

also analyzed as standards of comparison. Other methods of dating will be discussed in a later section of this chapter.

## THE SOCIAL SCIENCES

The solution of quantitative problems in psychology, education, sociology, government, economics, archaeology, history, and other social sciences involves the use of a number of mathematical methods. In psychology and related fields, mathematical techniques are helping experts unveil the mysteries of the human mind. Efforts to determine the *modus operandi* of the brain have disclosed a similarity between its operation and that of modern electronic computers. Because some, if not all, of the functions of the brain are electrical in nature, certain physical laws which can be stated mathematically are applicable.

In other areas of psychology and in education, statistical techniques are widely employed. For example, the norms for standardized objective tests of personality, intelligence, aptitude, scholastic achievement, etc. are statistically constructed. Data describing the performance of appropriate population samples are compiled and tested mathematically for validity and reliability. These results are used as measuring instruments of the performance of individuals given the test. Although by no means infallible, diagnostic and achievement test results are often useful indices of the psychological and educational adjustment of individuals.

Statistical techniques, along with more subjective case-study methods, are also applied to problems involving the behavior of societies and of units and groups within societies. Polls, for instance, developed and conducted under the direction of sociologists, are used to determine societal patterns and to measure and predict trends and changes. Industry and government, in turn, use these statistical data as bases for judgments.

In a society as complex as ours, officials of local, state, and federal governments use many kinds of statistical information in making decisions and long-range plans. The United States Bureau of the Census and other governmental agencies collect and compile facts touching virtually all phases of our national and international life. Although sometimes misused through ignorance or deliberate intent, statistics may help lawmakers decide and plan intelligently for the collection and disbursement of government money for industry, agriculture, scientific research, education, military security, health, welfare groups, etc.

One illustration of a problem of national concern is the status of education in our country. Are our children being well educated? How can we determine and judge the efficiency of our system? What can we do to better it? Although statistics cannot give the answers, objective consideration of these questions would be impossible without factual data. Using money expended as one of many criteria of the adequacy of an educational system, some state and local governments have increased expenditures for education partially on the basis of statistical information such as is given in Figure 7-14.

The problems of determining voting power in government bodies, corporate structures, etc., have in recent years been studied from a set theory approach. In large corporate structures where there is not a one-to-one correspondence between the number of voters and the

New York	\$869*	Ohio	\$503
Alaska	718	Kansas	495
New Jersey	662	Vermont	475
Wyoming	661	Utah	473
Connecticut	627	Missouri	471
Illinois	612	New Hampshire	469
Oregon	612	Louisiana	468
California	603	South Dakota	466
Montana	580	North Dakota	460
Rhode Island	576	Florida	458
Minnesota	573	Texas	450
Maryland	566	Nebraska	423
Pennsylvania	565	Virginia	415
Wisconsin	557	Oklahoma	411
Washington	556	Maine	410
Delaware	548	Idaho	388
Michigan	545	Georgia	384
Nevada	535	North Carolina	379
Colorado	530	Arkansas	376
Massachusetts	530	West Virginia	367
Arizona	524	Kentucky	364
New Mexico	524	Tennessee	361
Hawaii	513	Alabama	356
Indiana	512	South Carolina	350
Iowa	503	Mississippi	318

\* Data from the Department of Health, Education, and Welfare

FIGURE 7-14. Expenditures per pupil in average daily attendance for public elementary and secondary day schools: 1965-66.

number of votes to be cast (some members possessing more shares and/or proxies, and thus, more votes than others), such problems can become quite complex. Defining "winning coalition" as any subset of members of a voting body in which enough votes are possessed to carry a measure, and "blocking coalition" as any subset of members possessing enough votes to keep a measure from carrying, political strategists can precisely predict the various sets of elements (members) whose support will insure the success or failure of a measure.

Most areas of economics use mathematics freely in correlating and interpreting data, for stating certain laws or rules of finance, and in problems of design. In the field of finance, in addition to accounting techniques using elementary arithmetic, many other techniques and definitions utilize mathematics. The rules of compound interest and the methods for depreciating assets both make use of advanced algebraic methods.

The branch of economics which is concerned with industrial management is making an increasingly greater use of mathematics. One of the problems of modern industry is to design its production facilities to yield the greatest profit from the manufacture and sale of goods. A very new branch of mathematics, linear programming, is being used in this economic phase of industry.

A field closely allied with history (and with the earth sciences) is *archaeology*. The discovery and dating of artifacts help experts to rebuild the history of the earth and its peoples. Two of the more recent methods of dating artifacts, both employing mathematics, are the "Carbon-14" and the "tree-ring" methods.

The Carbon-14 method, like the half-life dating described earlier in this chapter, is based on the rate of decay of radioactive material and is useful in determining the age, up to 25,000 or 30,000 years, of organic materials such as wood, bone, and fossilized pollen. (Since the half-life of Carbon-14 is approximately 5,600 years,  $\frac{1}{4}$  of it remains after about 11,000 years, and almost all of it disappears in 25,000 to 30,000 years.) All living plants and animals take in radiocarbon (Carbon-14) from the air, where the carbon isotope is formed by the action of cosmic rays (streams of atomic particles) on atoms of nitrogen. In the living cells the carbon is combined with oxygen to form carbon dioxide. (The proportion of radiocarbon to ordinary carbon in the air we breathe is only one to one trillion.) Although constantly disintegrating, the supply of radio carbon in a living organism is continually being renewed. At death, the organism ceases to take in the material; however, decay of the element continues at its constant rate. When dating an object, the sample, after being reduced to pure carbon, is measured for

radioactivity by means of a Geiger counter. Since the half-life standard is given in terms of a pure sample of definite gram weight, mathematical adjustments must be made before the age of the object may be determined.

It will be noted that the Carbon-14 method is both more limited in time span and more precise than the uranium-to-lead method. An instrument of time measure still more limited and more precise is the tree-ring method of dating. This method, developed about 1920 by an American astronomer, is based on the fact that each ring of a tree both represents and climatically "describes" a particular year in a certain area. Years of drought, for example, are described by thin rings; years of good growing conditions, by wide rings. Further, in the same year the same general pattern occurs in all trees of the area. By matching rings found in successively older pieces of wood, time scales of several hundred years have been pieced together for various regions. The age of a structure using wood, for example, may be determined by matching the rings of its wood with those of the master tree-ring scale.

Besides enabling historians and others to fix past events in time and space, mathematics helps them predict future occurrences. Some events can be predicted on the basis of probability—on statistical surveys of history. This is what is meant by the old saying: "History repeats itself."

In its larger sense, *history* itself may be said to include all of the mathematics, pure and applied, created by man. The use made of Einstein's relativity theory in the design of atomic weapons utilized by the Allies in World War II testifies to the sometimes dramatic role of mathematical research in influencing history's course.

We have seen how the development of numeration systems and mathematical techniques has been necessitated by man's need to define, communicate, and explore the quantitative aspect of his environment. Partly as a result of mathematical pioneering, this environment has become increasingly complex. If scientific, technological, and humanitarian progress in the United States and in the world are to continue, the challenge of training able young people meticulously and comprehensively in mathematics must be met.

## V. COMPUTERS AND MODERN TECHNOLOGY

The concept to be developed in this section is:

*Modern technology is increasingly using mechanical means of performing arithmetic.*

We are entering an age in which technology is presenting us with arithmetic problems of enormous complexity. If the answers to such problems are to have significance, we cannot wait years for solutions. High-speed computers have cut the time for solving these problems from years to days, hours, or minutes.

Comparisons of speed are always relative, but one example may serve to set the stage. A problem which would take (and actually has taken) a trained mathematician seventy hours to work with the aid of an electric desk calculator can be worked (and has been worked) on a moderate-speed electronic computer in eighteen minutes. On a computer which is today considered merely high speed, the same problem can be worked in well under one minute; and the slow part of the operation is not the computation, but the reading of the problem and the writing of the answer.

With such an age not just around the corner, but actually here now, no teacher of arithmetic should be entirely unaware of the uses for and trends in the development of computers.

## ARITHMETIC COMPUTATION

With the dawn of recorded history man began recording facts by using symbols to represent numbers. As he increased his ability to describe the world in which he lived with numerals, he needed new methods of calculating or computing. His existing numeral systems, however, were poorly suited to calculation. As time progressed, he modified these old systems of numeration, or abandoned them entirely in favor of better ones. With better systems of numeration he was able to devise algorithms which enabled him to compute accurately and rapidly. The trend was away from the crude mechanical devices which had been used as computational aids, and toward more knowledge and ability on the part of the person performing the calculations.

As society developed further, the needs for more complex calculations increased, and the pendulum swung back toward the use of mechanical devices (currently, for example, slide rules, desk calculators, etc., used by trained operators). This was inevitable because of the rapid rise in the number of these kinds of calculations. This trend, however, should discourage neither the teacher nor the pupil from devoting full attention to arithmetic education, for no machine performs operations not encompassed by its design.

Following this trend, our horizons have broadened to such a point that we can solve mathematics problems of a complexity that our

grandfathers could not even have conceived. Only in the last two decades have we finally been able to loosen (if not to break) the shackles which have bound us to slow-speed computing devices and now are beginning to solve many of the problems which we are able to envision.

## BRIEF HISTORY OF COMPUTERS

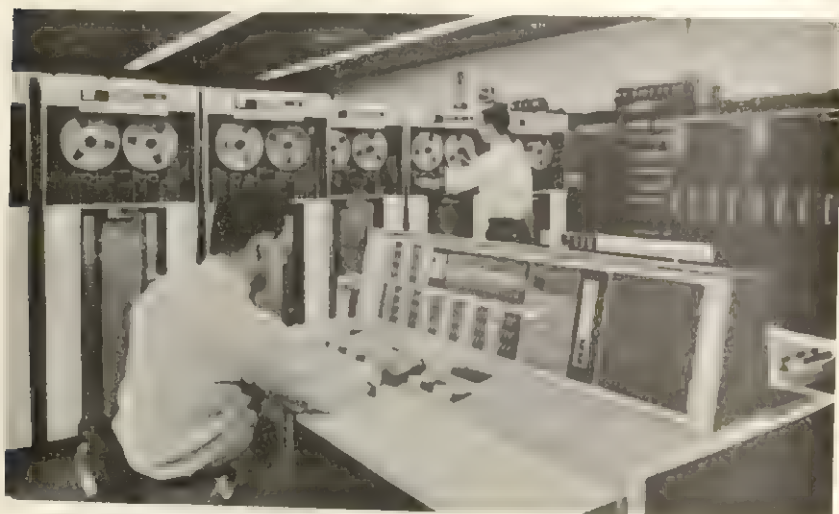
The earliest mechanical devices for performing calculations mechanically were probably the abaci of various designs which date back to well before the Christian Era. There was little evolution in mechanical computation until about 1600 A.D. when John Napier invented a device, subsequently called "Napier's Bones", which facilitated multiplication.



Courtesy of International Business Machines Corporation  
*Napier's Bones*

It was followed, some twenty years later, by the invention of the slide rule, which would perform both multiplication and division. (A slide rule is a type of analog computer because it sets up analogies between lengths and numbers rather than working with the individual digits of numerals representing numbers.) In 1642 Blaise Pascal devised the first adding machine, a model which was copied, modified, and improved during the next 200 years. In the nineteenth century, Charles Babbage designed several machines to compute and print tables auto-

matically. In 1875 Frank Baldwin invented a calculator which would perform all four of the fundamental arithmetic processes. The electric desk calculators of today are, for the most part, modifications and improvements of the Baldwin model.



Courtesy of International Business Machines Corporation  
*IBM 7090 Data Processing System*

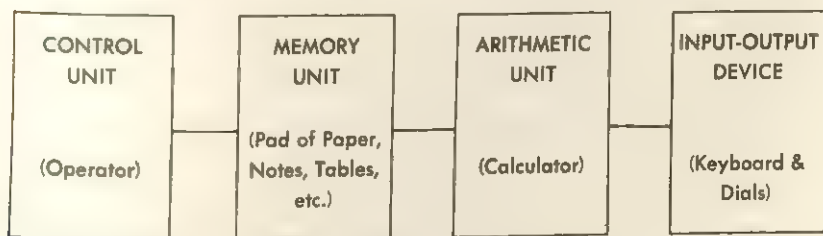
In 1943 the first electronic calculator was built, and by 1944 it was in operation at Harvard University. World War II and its problems hastened the developmental stages of electronic computers so that by 1953 various organizations had several models in production. The computers of the middle 1950's are considered to be of medium speed. In the late 1950's multi-million dollar high-speed computers of great complexity began to appear. The designs of these electronic computers differ in many respects; however they do have many features in common. They all have memory devices in which data and intermediate results are stored. Early computers used visible spots on the face of an oscilloscope (similar to a TV picture tube) to record information, but such storage of data was subject to too many difficulties, so better means of storage were developed. Later models stored data on magnetic tape (similar to an ordinary tape-recorder tape), on magnetic cylinders, and on magnetic discs. With the advent of the high-speed computers, a new type of device was introduced. It was a matrix, or arrangement, of ferrite beads which could be magnetized or demagnetized. Today,

a modern computer may use a combination of these various memory media.

From the above facts about the memory devices, and mentioning the fact that where the older computers used vacuum tubes, the newer ones use transistors, it should be apparent that only two states of being are possible in any component of the computer: switches can be open or closed; vacuum tubes may be operating (passing electrons through) or not operating; magnetic spots or beads may be magnetized or not magnetized. Consequently, the internal arithmetic operations of most computers are carried on in base two (in which any number can be represented by a combination of only the two symbols 0 and 1, each symbol representing one of the two states of being). Thus the base two numeral system, which was until recently only a mathematical curiosity, has found its place in everyday modern arithmetic.

## DESCRIPTION OF COMPUTERS

While the word computer in modern usage usually implies electronic computer, at this stage we will use it to denote any sort of calculating device. Ordinarily the desk calculator is only one of several components which must work together to solve a problem. The entire complex might be said to be composed of the operator, the calculator, a pad of paper and pencil for recording intermediate and final data, and tables, notes, etc. on which data are stored for present and future reference. The following diagram illustrates a comparison between this complex and that of an electronic automatic computer. The units are interconnected, but for many practical purposes they are independent units, with independent purposes and functions.



The operator of a desk calculator controls the type and sequence of operations, and makes logical decisions regarding initiating and terminating operations. He makes reference to the memory unit (tables, etc.)

for data; he stores intermediate data in his head or on paper; he pushes keys or buttons to put information into the arithmetic unit; and he reads dials to obtain information from the arithmetic unit.

The main difference (aside from design, speed and other physical factors) between a desk calculator and an automatic electronic computer is that the role of the operator is changed. Both have essentially the same four components; however in the electronic computer, the control unit component is not a person. The sequence of steps and the logical decisions which the computer is to follow must be put into its memory before it can follow the necessary sequence of steps (program) automatically. It is the introduction of this automatic following of a sequence of operations which constitutes the real progress made in computation in the last two decades. The operator is not discarded, though, since he must write the program which the computer is to follow, and cause the computer to start the sequence. Furthermore, he must be able to interpret the answers the computer generates.

It has already been suggested that a slide rule is an analog computer. Many other types of analog computers, designed for special purposes, exist. A digital computer, on the other hand, operates by using the individual digits of the numerals which represent numbers. Digital computers vary somewhat in design, but, for the most part, they are able to perform only additions and subtractions (in many computers, subtractions are performed by the addition of complements). Multiplications are performed by repeated additions and shifting (the shifting comparable to that used in our modern algorithm), and divisions are performed by repeated subtractions and shifting. In addition to performing the four fundamental arithmetic processes, modern electronic digital computers can make a small number of logical decisions (for example, they can test to see if a number is zero or if it is negative). They can follow a predetermined program automatically, but they cannot think. They cannot teach themselves to perform operations that they have not been programmed or wired to perform. They will never be able to supplant the human brain in the realm of creativity. They *can*, however, perform complex arithmetic operations with greater accuracy and speed than a human operator could ever hope to achieve.

## PROGRAMMING FOR DIGITAL COMPUTERS

In the early days of high-speed computation, the programmer was obligated to organize his problem into a large number of small steps,

and then write instructions for these steps in some form which the computer could recognize. These instructions usually took the form of a numerical code. For example, the first two digits of the numeral might represent an operation to be performed (say, multiplication), the next four digits might represent an address (location of the multiplier), and finally, the next four digits might represent the address of the next instruction to be executed. A program of medium length might contain several hundred such numerically coded instructions.

As the complexity of the problems to be worked increased, the programmer's time was more and more occupied with the mere bookkeeping operation of being sure all necessary steps were included and that no improperly coded instruction or address had been given. It became apparent that this type of mechanical work was just the sort of thing the machine could be programmed to do, so new systems were devised which allowed the programmer to describe his operations by mnemonics rather than by numerals; but he still had to describe every minute step in his arithmetic processes.

Only in recent years has it been recognized that many arithmetic processes repeat the same set of steps every time the process is done. The latest development, then, is to allow the programmer to write his problem in words and symbols and equations, much as he would describe it algebraically, and the machine translates these into a program it can follow. This allows the programmer to write a program for a fairly long or complex operation in only a few steps and then depend on the machine to perform the mechanics of reorganizing the operation into small steps which it can execute.

## CAPACITY AND SPEED

One of the aspects of machine computation which never fails to amaze the uninitiated is the speed at which modern high-speed computers operate. A medium-speed computer can multiply 2 ten-digit numbers (numbers represented by numerals of ten decimal digits) in about one one-hundredth of a second. It has already been indicated that the modern medium-speed computer is at least 300 times faster than a good desk calculator. The modern high-speed computers can execute a given sequence of commands from 50 to 1,000 times as fast as the intermediate-speed computers. Ultra-high-speed computers can execute approximately 1,000,000 commands per second.

Almost any modern computer has a memory size of at least 2,000

ten-digit numbers. Of course, some of these numbers are numerically coded commands and some are data. The largest of the high-speed computers in use today have up to 250,000 memory cells, each cell able to store a 15 to 20 digit number.

The conclusion which may be drawn from this information is that both extreme accuracy and high speed are available in today's computers. Of course, because of inherent arithmetic difficulties, the computer may give answers which are accurate as far as the computer is concerned, but which are just so much rubbish as far as actual accuracy is concerned. For example, if a measured quantity which has a small error because of inaccuracies of measurement is multiplied by a large multiplier, the answer the computer obtains may be perfectly accurate as far as the multiplication is concerned, but quite worthless as far as accuracy of measurement is concerned. The error of measurement is magnified in the multiplication process.

## ELEMENTS IN FUTURE COMPUTERS

What can we look for in the near and distant future in the field of computation? Here are some of the features which are almost sure to appear.

1. *Miniaturization.* One of the early electronic computers occupied a 30 by 50-foot room. Some computers today with equal or greater capacity and speed are not much larger than an ordinary desk. With the use of transistors and other miniaturized components, computers undoubtedly will become not much larger than desk calculators.

2. *New components.* Vacuum tubes have given way to transistors. In the near future, design changes will allow inclusion of tunnel diodes, semiconductors, printed circuits. All of these may lead to increased efficiency, reliability, and freedom from breakdown.

3. *Speed and capacity.* The computers now coming off the drawing boards and entering the production stage will be able to perform several million operations per second. They will possess memory and storage devices which will have immediate accessibility (e.g., no waiting for magnetized tape to wind and rewind) and which will contain up to 1,000,000 or more cells.

4. *Potential.* Still in the theoretical stages is an idea which involves the elimination of many of the wires which join the electrical components. If these components can be joined to a conductor, and the

molecules polarized, then current could be passed from one lead to another by means of these polarized molecules. With the elimination of resistance and with the miniaturization and introduction of new type components into computers, the only limit to speed is the speed of light. Engineers are working on the problem of approaching this speed in the functioning of computer components.

Computers are becoming more and more expensive, but their impact is being increasingly felt on our daily lives. All teachers of mathematics should be acquainted with this fascinating and growing field which is so closely allied with arithmetic.

## EXERCISES

1. Draw a picture of a house using one-point perspective; draw another picture using two-point perspective.
2. Find the frequencies of vibration of all eight tones of the octave C — C'. Show that the vibration fractions for major third, perfect fourth, and perfect fifth do not represent the exact ratios of the frequencies.
3. Approximately how many miles constitute a light year?
4. Approximately how long would it take for light to travel along the diameter of Earth's orbit?
5. Why does the moon appear to be larger at moon-rise than it does three or four hours later?
6. Find the physical definition of "work." How much work will be done in lifting a 100 lb. weight 6 feet? Discuss the meaning of "horsepower."
7. One mole  $H_2O$  (water) contains 2 gram-atoms of hydrogen and 1 gram-atom of oxygen. The weight of 1 mole is 18 grams. The weight of hydrogen in 1 mole is  $2 \times 1 = 2$  grams. Approximately what percentage of water (by weight) is hydrogen, and what percentage is oxygen?
8. Read and report on the discussion of voting coalitions found in Kemeny, Snell, and Thompson: *Introduction to Finite Mathematics*.
9. Discuss other uses of mathematics in biology, economics, geology, etc.
10. Construct a set of Napier's Bones and learn to use them.
11. Read and report on the use of a slide rule.
12. Find three uses of computers not considered in this chapter and report on your findings.

13. Select five pages from a geography or science textbook. Count the number of (1) definite and (2) indefinite quantitative terms found in the text.
14. The Jones family took an afternoon ride. For the first hour their average speed was 30 miles per hour. For the second hour it was 60 miles per hour. Was their average speed for the two hours 45 miles per hour, or more or less than 45 miles per hour?
15. On another day the Jones family drove 45 miles at an average speed of 30 miles per hour, then returned at an average speed of 60 miles per hour. Was their average speed for the whole trip more than, less than, or equal to 45 miles per hour?
16. A solution of tincture of iodine contains  $2\frac{1}{2}\%$  iodine by weight. How much iodine is needed to prepare 1200 grams of the solution?
17. A mixture of alcohol and water contains 30% alcohol, by volume. To make the mixture, water was added to 15 pints of alcohol. How much water was added?
18. (a) Use the formula  $F = \frac{9C}{5} + 32$  to find the Fahrenheit temperature corresponding to a Celsius (centigrade) temperature of (1)  $72^\circ$ , (2)  $50^\circ$ , (3)  $300^\circ$ .  
 (b) Use the same formula to find the Celsius (centigrade) temperatures corresponding to these Fahrenheit temperatures: (1)  $100^\circ$ , (2)  $212^\circ$ , (3)  $72^\circ$ .
19. A ball is allowed to roll down an inclined plane. The distance, in centimeters, it rolls in 1, 2, 3, or 4 seconds is recorded in the following table. The letter  $t$  stands for the number of seconds, and the letter  $d$  stands for the number of centimeters the ball rolls (the distance from the top of the inclined plane).

$t$	0	1	2	3	4
$d$	0	4	16	36	64

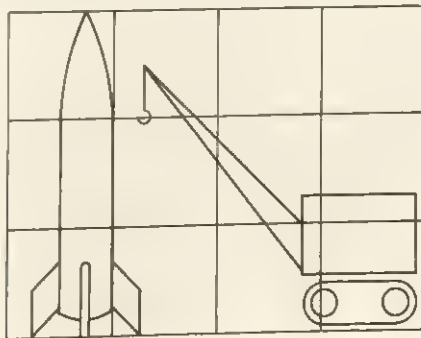
Draw a broken line graph of these data. Let the horizontal axis of the graph represent time, the vertical axis distance.

20. (a) From your graph, estimate the distance the ball will roll in  $3\frac{1}{2}$  seconds.  
 (b) Estimate how many seconds it will take the ball to roll 100 centimeters.

21. Can you find a formula which is satisfied by the data in exercise 19?
22. The children in a sixth grade science class were investigating the effect of chemicals on plant growth. After growing bean plants for 2 weeks they measured the lengths of each of 15 plants grown under each of three conditions. The following table gives the results.

TEST	CHEMICAL	LENGTH IN CENTIMETERS
A	no chemical	18, 16, 21, 17, 23, 12, 20, 18, 15, 19, 21, 19, 18, 17, 18
B	salt	2, 7, 9, 8, 3, 6, 5, 7, 4, 2, 1, 10, 7, 8, 5
C	commercial fertilizer	22, 16, 30, 28, 27, 21, 25, 29, 30, 20, 19, 24, 27, 28, 26

- (a) What was the average length of the plants in Test A? Test B? Test C?
- (b) What tentative conclusion can be drawn from this data?
- (c) Name some factors which might cause this conclusion to be invalid or have doubtful validity.
23. A meteorologist knows that a cold front is 150 miles from his observation station and that it is moving toward the station at an average rate of 18 miles per hour. If conditions do not change, in how many hours should he expect the front to reach the station?
24. One way to enlarge a picture is to use scale drawing. First a grid is superimposed on the picture, and then the corresponding points are found and drawn on a grid of larger mesh. Draw a grid with vertical and horizontal lines 1 inch apart on your paper. Using this grid, enlarge the simple picture below.



25. In Exercise 24, your linear dimensions are what multiple of those in the illustration? The area of your figures is what multiple of the area in the illustration?
26. If a modern computer can perform an average of 80,000 arithmetic operations in one second, the average time per operation is how many millionths of a second?

## Extended Activities

1. Read and report on the article "Mathematics of Population and Food" in *The World of Mathematics* by James R. Newman.
2. Find, read, and report on other books, chapters, or reports on the broader applications of mathematics to the world in which we live.

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## Chapter 8

Concepts to be developed in this chapter are:

1. *Mathematical recreations can be used to enhance the child's interest in mathematics.*
2. *Mathematical understandings can be broadened through the use of appropriate recreations.*

## Extending Understanding and Interest with Mathematical Recreations

The story is told of an ancient king of India who wished to reward his grand vizier for inventing and teaching him the game of chess. The vizier's request seemed quite humble: a single grain of wheat for the first square of the chessboard, two grains for the second, four for the third, eight for the fourth, sixteen for the fifth, and so on, doubling the number of grains in each succeeding square until all 64 squares had been filled. Promising to grant this modest request, the king ordered a bag of wheat brought before the throne. As the counting proceeded, his pleasure changed to alarm, then to astonished rage. It became increasingly apparent that the entire wheat crop of India would be insufficient to fulfill the vizier's request! Indeed, calculation discloses that the clever minister had placed the king in his debt for the sum of 18,446,744,073,709,551,615 grains of wheat, or approximately the world's wheat crop for almost two thousand years!

Thus, for centuries, numbers have amazed, tricked, delighted, and baffled man. As old riddles are solved, new ones arise to tantalize and to challenge.

Children, along with the world's greatest mathematicians, should have the opportunity to play, as well as to work, with numbers and their representations. Many recreational materials may sharpen not only interest in mathematics, but insight into basic mathematical concepts, techniques of problem solving, and the ability to construct and use models. (In fact, a branch of mathematics, number theory, has developed chiefly as an outgrowth of early number puzzles and games.) Other materials are useful primarily in providing interesting practice in computation.

In this chapter we shall take a brief view of mathematical recreations, sampling a few materials from various areas. As in choosing any mathematical activities, when selecting recreational materials, the teacher needs to give thoughtful consideration to the needs and abilities of his students. Some activities he may wish to present to the entire class; others, to individuals or small groups for whom they are appropriate.

## MAGIC FIGURES

Magic squares, circles, triangles, etc., are among the most fascinating of mathematical recreations and are adaptable to many levels and purposes. Their "magic" usually lies in the fact that the numbers represented in each horizontal, vertical, and diagonal column produce the same sum. "Subtraction" and "multiplication" figures may also be devised.

Children may test magic figures to determine whether, and to what extent, they are magic. In Figure 8-1, for example, we find that the sum of each column, each row, and each diagonal is 15.

(1) The figure below is magic in three ways. Can you find the ways?<sup>1</sup>

8	1	6
3	5	7
4	9	2

FIGURE 8-1

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<sup>1</sup> Solution to this and other numbered materials, pp. 148-150.

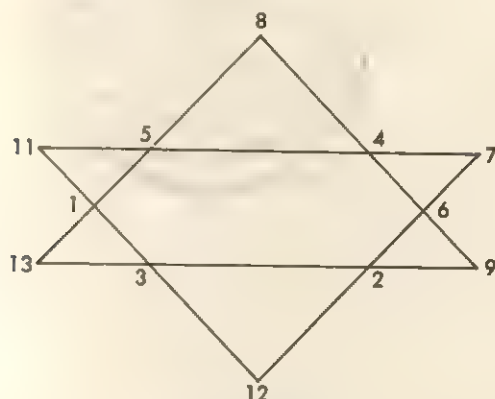


FIGURE 8-2

The problem may consist of constructing a figure, or of filling in missing numbers, as the two examples in Figure 8-3 illustrate.

The understanding that the same operation may be performed on each term of an equation without altering the relationship of the terms may be promoted by having children first check a square for "magic," then add, subtract, multiply, or divide each term by the same number and recheck (Figure 8-4).

Children enjoy constructing magic squares and exchanging them with classmates for checking. Fifth and sixth grade students may formulate and arrange sequences of squares so that, through the

(2) Place the numerals 1 through 9 so that each side of the triangle totals 17:



(3) Fill in the missing numbers to make a magic square:

$2\frac{1}{2}$		2
	$3\frac{1}{4}$	
$1\frac{1}{2}$		1

FIGURE 8-3

Is this a magic square?

32	4	24
12	20	28
16	36	8

(4) Divide each number by 4. Is this a magic square?


FIGURE 8-4

proper choice of multiples, divisors, etc., the original square is repeated as the last in the sequence.

Two exceptionally "magic" figures are known as the Melancholia square and the IXOHOXI square.

The Melancholia received its name from the title of a painting by Dürer (1514) in which it was shown. It is reproduced in Figure 8-5.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

FIGURE 8-5

A number of its "magic" properties are:

- (a) the sum of each horizontal, vertical, and diagonal column is 34;
- (b) the sum of the four center squares, of the four corner squares, and of the squares 3, 2, 15, 14, and 5, 9, 8, 12 is 34;
- (c) the sums of both the upper half and lower half are 68;
- (d) the sums of the right half and of the left half are 68;
- (e) the sums of columns 1 and 3 and of columns 2 and 4 are 68;
- (f) the sums of rows 1 and 3 and of rows 2 and 4 are 68;
- (g) the sum of the squares of the numbers contained in each of (d), (e), and (f) is 748;

8818	1111	8188	1881
8181	1888	8811	1118
1811	8118	1181	8888
1188	8881	1818	8111

FIGURE 8-6

- (h) the sum of numbers in the diagonals and of those not in the diagonals is 68; the sums of their squares is 748; and the sums of their cubes is 9248.

Like its name, the IXOHOXI square (Figure 8-6) is a novelty in that it can be read right side up, upside down, and both of these ways in the mirror. The sum 19998 may be found by adding each vertical, horizontal, and diagonal column; the center square; the corner squares; the inside squares of the top and bottom rows; the inside squares of the first and fourth column; and each small square composed of four adjacent squares—right side up, upside down, right side up in the mirror, and upside down in the mirror.

Children should be able to discover many of the “magic” properties of these latter two squares. Those properties not found by the students may be proposed by the teacher for the pupils to test.

## CHECKS AND SHORT CUTS

An interesting supplementary method of checking computation, appropriate for upper-grade children with knowledge of conventional methods, is “casting out nines.” This check has as its basis the fact that any integer divided by 9 has the same remainder as the sum of its digits divided by 9. (Stated mathematically, a number is *congruent* to the sum of its digits modulo 9. The relation “is congruent to” is symbolized by  $\equiv$ .) For example, dividing 24,796 by 9 in Figure 8-7, we find that the excess of nines, or remainder, is 1. By adding the digits of 24,796, then dividing by 9, we obtain the same remainder. Or we may add the digits of the integer obtained by the earlier addition and continue this process until a number less than nine, the excess, is reached.

$$\begin{array}{r}
 2755 \\
 9 \overline{) 24,796} \\
 \underline{18} \phantom{00} \\
 67 \phantom{00} \\
 \underline{63} \phantom{00} \\
 49 \phantom{00} \\
 \underline{45} \phantom{00} \\
 46 \phantom{00} \\
 \underline{45} \phantom{00} \\
 1
 \end{array}$$

$$2 + 4 + 7 + 9 + 6 = 28$$

$$28 \div 9 = 3 \quad r \ 1$$

$$2 + 4 + 7 + 9 + 6 = 28$$

$$2 + 8 = 10$$

$$1 + 0 = 1 \text{ (excess of nines)}$$

$$24,796 \equiv 28 \equiv 10 \equiv 1 \pmod{9}$$

FIGURE 8-7

Thus, it is possible to check *addition* by determining the excess of nines in each addend, finding the total, and checking this total against the excess of nines in the sum, as in Figure 8-8. However, casting out nines is not an absolute check and should be used judiciously.

$92784 \equiv 9 + 2 + 7 + 8 + 4 =$	$30 \equiv 3 + 0 =$	3
$32671 \equiv 3 + 2 + 6 + 7 + 1 =$	$19 \equiv 1 + 9 = 10 \equiv 1 + 0 =$	1
$59242 \equiv 5 + 9 + 2 + 4 + 2 =$	$22 \equiv 2 + 2 =$	4
$63489 \equiv 6 + 3 + 4 + 8 + 9 =$	$30 \equiv 3 + 0 =$	3
$248186 \equiv 2 + 4 + 8 + 1 + 8 + 6 = 29 \equiv 2 + 9 = 11 \equiv 1 + 1 = 2$		$\overline{11} \equiv 1 + 1 = 2$

↙ Check ↘

FIGURE 8-8

A shorter method involves casting out nines and digits whose sum is nine, before actually adding the digits of the number. Thus, in finding the excess of nines of the first addend in Figure 8-8, we might initially discard the first three digits (9, 2, 7), add  $8 + 4$ , and subtract 9 from the 12 obtained, leaving the same remainder, 3.

To check multiplication, after casting out nines we multiply the excesses as shown in Figure 8-9.

Subtraction is checked by casting out nines and subtracting the remainders. If the subtrahend's remainder is larger than that of the minuend, we may add 9 to the minuend's (Figure 8-10).

In Figure 8-11 division is checked by multiplying the excess of nines in the divisor and quotient and adding the excess of the remainder.

It will be noted that the operations performed in checking division by this method are the same as those employed in checking by the

<del>227</del> $\times \quad \cancel{9}\cancel{8}$ <hr style="width: 100px; margin: 0;"/> 2376 2673 <hr style="width: 100px; margin: 0;"/> 29106	<div style="text-align: right; padding-right: 10px;">Check</div> $297 \equiv 0$ $98 \equiv 8$ <hr style="width: 50px; margin: 0;"/> $0$
<del>227</del> 106	<div style="display: flex; align-items: center; justify-content: center;"> <div style="text-align: right; padding-right: 10px;">0</div> <div style="text-align: center;">← Check →</div> <div style="text-align: left; padding-left: 10px;">0</div> </div>

FIGURE 8-9

conventional method, and that multiplication and subtraction may also be checked by their inverse processes.

"Casting out elevens" is a somewhat similar method of checking computation which, because of space limitations, will not be detailed here.

<del>976245</del> $- \cancel{2}\cancel{7}\cancel{9}260$ <hr style="width: 100px; margin: 0;"/> 696985	<div style="text-align: right; padding-right: 10px;">Check</div> $976245 \equiv 6 \equiv 6 + 9 = 15$ $279260 \equiv 8$ $696985 \equiv 25 \equiv 2 + 5 = 7$
	<div style="display: flex; align-items: center; justify-content: center;"> <div style="text-align: right; padding-right: 10px;">15 -8 <hr style="width: 50px; margin: 0;"/>7</div> <div style="text-align: center;">← Check →</div> <div style="text-align: left; padding-left: 10px;">7</div> </div>

FIGURE 8-10

## FINDING THE MISSING NUMERALS

The mathematical concept of the variable, the number or numeral for which a letter symbol is placeholder, is important for students to

$80$ $37 \overline{) 2964} \equiv 12 \equiv 3$ $296$ <hr style="width: 50px; margin: 0;"/> $4 \text{ r.}$	<div style="text-align: right; padding-right: 10px;">Check</div> $80 \equiv 8$ $37 \equiv 10 \equiv 1$ $8 \times 1 = 8$ $8 + 4 = 12 \equiv 3$
	<div style="display: flex; align-items: center; justify-content: center;"> <div style="text-align: right; padding-right: 10px;">3</div> <div style="text-align: center;">← Check →</div> <div style="text-align: left; padding-left: 10px;">3</div> </div>

FIGURE 8-11

<p>(5)</p> $  \begin{array}{r}  a34 \\  567 \\  4bc \\  201 \\  \hline  d497  \end{array}  $	<p>(6)</p> $  \begin{array}{r}  a46 \\  \times 35b \\  \hline  14c2 \\  3d30 \\  2238 \\  \hline  2e2f9g  \end{array}  $
<p>(7)</p> $  \begin{array}{r}  935284 \\  - abcde \\  \hline  f35285  \end{array}  $	<p>(8)</p> $  \begin{array}{r}  a0b \\  23 \overline{) 69c6} \\  \underline{69} \phantom{00} \\  4d \\  \underline{46} \\  \phantom{00}  \end{array}  $

FIGURE 8-12

grasp. A recreational device which may promote development of this idea along with computational skill is the "missing numeral" puzzle. Such puzzles may be varied in difficulty and may involve any fundamental operation. In Figure 8-12 examples of addition, multiplication, subtraction, and division puzzles are given.

## NUMBER GAMES

Number games which provide practice in counting, recognizing groups, and in the basic addition, subtraction, multiplication, and division facts are described in many texts and supplementary materials for teachers. Such devices, carefully chosen and judiciously used, may enliven interest and add fun to the mathematics program. Frequently, after introducing a game to the class or a group, teachers place the necessary materials at the activity center, available for free-time use, to be replaced by others as interests wane and new topics are taught. A few examples follow.

*Odd or Even?* This game gives practice in identifying odd and even numbers. Each of two players has ten small objects (beans, grains of corn, etc.) At his turn, each child holds his objects behind his back and

	1	2	3	4	5	6
4						
5						
6						
7						
8						
9						

FIGURE 8-13

divides them between his hands. He then extends one hand, closed, to the other player, and asks, "Odd or even?" If his opponent guesses correctly, he gets the objects; if not, the asker keeps them. After five turns each, the player with the most objects wins.

*Number Checkerboard.* This two-player game is played with a checkerboard arrangement as in Figure 8-13 and may be varied as to the number of squares, the numbers used, and the operation to be performed. Each player has a number of markers distinguishable from those of the other player by color, etc. To begin the play, one player points to a square. His opponent must then give the correct sum, difference, product, or quotient of the number pair represented by the square. If he answers correctly, he places his marker in the square and then takes his turn asking a question. If he misses, the other player may give the correct answer and place his marker on the square, then take another turn. The player who places the most markers wins.

*Number Bingo.* A group or the entire class may play this game. Each player has a different bingo card, which may have addition, subtraction, multiplication, or division problems written in the squares (Figure 8-14). As numbers are called, each player looks for a problem on his

$81 \div 9$	$54 \div 7$	$9 \div 3$
$32 \div 8$	Free	$54 \div 6$
$56 \div 7$	$63 \div 9$	$25 \div 5$

FIGURE 8-14



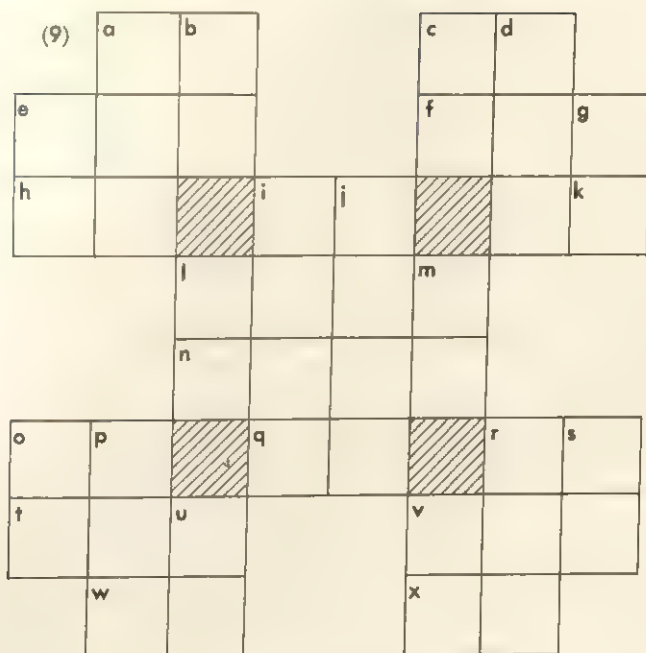


FIGURE 8-16

## ACROSS

- a. Jane's aunt sent her  $\frac{1}{8}$  ounce of perfume which cost \$3.25. How much would 1 ounce cost?
- c. What is the perimeter of a rectangle 12" by 9"?
- e. The Young family left their home at 8 A.M. and arrived at their summer camp at 6 P.M. They stopped 45 minutes for lunch and made three 15-minute stops for cold drinks. If they averaged 52 miles an hour during the time they drove, how far was the camp from their home?
- f. If  $\frac{3}{4}$  of a roll of carpet costs \$261.00, how much does the whole roll cost?
- h. Mary's mother is 3 times as old as Mary. In 12 years, Mary will be twice as old as she is now. How old is her mother?
- i. Three hundred quarters are how many dollars?
- k. Jack deposited in the bank the \$64.00 that he made mowing lawns during the summer. He withdrew  $\frac{3}{8}$  of the money to buy Christmas presents. How much did he have left in the bank?

- i. Mr. Black bought a ranch for \$278,190. If the land sold for \$90.00 an acre, how many acres did he buy?
- n. How would you represent the number eight in base two?
- o. A room of 144 square feet is 9 feet wide. How long is it?
- q. Joan planted 130 tulip bulbs. When they bloomed, 20% were yellow. How many were yellow?
- r. John was born in 1948. He will be eligible to vote in 19\_\_\_\_\_.
- t.  $64 \times 2 \div (8 \times \frac{1}{4}) + 4 \times 16 = N$
- v. Last year Mr. Montgomery drove 45,500 miles. How many miles did he average a week?
- w. Susan bought 8 yards of fabric at \$1.75 a yard for her recital dress. How much did the material cost?
- x.  $[(12 \div \frac{1}{2}) \times 4 + 12] \div 9 = N$

## DOWN

- a. The sum of 78, 112, and 56.
- b. The quotient of 6014 divided by 97.
- c. The difference between 7109 and 7066.
- d. The product of 488 and  $\frac{1}{2}$ .
- e. Jim saved \$10.00 to buy Christmas presents for his family. He chose gifts which cost \$3.97, \$2.10, and \$3.50. How much did he have left?
- g.  $2 \times 2 \times 2 \times 10 = N$
- i. The product of 389 and 18.
- j. Jill is checking a division problem. The divisor is 27; the quotient, 218; the remainder, 20. What is the dividend?
- l. Susan and her father took a 1829-mile trip in their new Italian car last summer. Susan kept the following record of the gasoline they bought: 10 gals., 9 gals., 7 gals., 9 gals., 6 gals., 5 gals., 8 gals., 5 gals. How many miles to the gallon did they average on the trip?
- m. The numeral which represents the number two in base two.
- o. The numeral which represents the number seven in base six.
- p.  $\frac{1}{3}$  of 1863.
- r.  $[(672 \div 2) \times 3 \times 6] \div 9 = N$
- s. A rectangular building has an area of 20,045 square feet and is 211 feet long. How wide is it?

- u. What two-digit number, when multiplied by the three lowest primes, gives the following products respectively: 168, 252, 420?

v.  $\left(\frac{9 \times 9}{9} \div \frac{9}{9}\right) \times 9 = N$

## MAGIC NUMBERS

Some numbers have been identified for centuries with mystery and magic. The number nine, for example, has numerous fascinating properties, many of which are attributable to its being the largest number preceding the most commonly used base. The products table of nine shows that as the tens digit of each successive product is increased by one, the ones digit is decreased by one with the result that the digits of each product have nine as their sum (Figure 8-17). This fact is not only interesting for children to discover in analyzing the table, but also can be an aid to remembering the nines products.

$2 \times 9 = 1$	$8 \rightarrow 9$
$3 \times 9 = 2$	$7 \rightarrow 9$
$4 \times 9 = 3$	$6 \rightarrow 9$
$5 \times 9 = 4$	$5 \rightarrow 9$
$6 \times 9 = 5$	$4 \rightarrow 9$
$7 \times 9 = 6$	$3 \rightarrow 9$
$8 \times 9 = 7$	$2 \rightarrow 9$
$9 \times 9 = 8$	$1 \rightarrow 9$

FIGURE 8-17

The following are other curiosities of nine which might be introduced as number tricks or puzzles:

- (a) Take any number. Change the order of the digits and subtract the smaller number from the larger number. When the digits of the difference are added, the sum is always some multiple of nine.

For example:

$$\begin{array}{r} 7425 \\ -4275 \\ \hline 3150 \end{array} \quad 3 + 1 + 5 + 0 = 9$$

- (b) From these numbers, select your favorite: 1 2 3 4 5 6 7 9. (Note that 8 is omitted.) Multiply the number you have chosen by nine; then multiply the product by the entire series above. The result will be a product composed of the number chosen.

Number chosen: 7

$$\begin{array}{r} 7 \times 9 = 63 \\ 12345679 \\ \phantom{12345679} 63 \\ \hline 37037037 \\ 74074074 \\ \hline 77777777 \end{array}$$

## BRAIN TWISTERS

Almost everyone delights in conquering a "brain twister," and excellent opportunities for the construction and use of models are afforded by many such problems. Frequently, the teaser's answer is next to obvious after opening the mind to an other-than-conventional conception of the situation, and its possible solutions. The problems below are representative of a few of the types which might challenge children to do clear, objective thinking.

- (10) A class wanted one gallon (four quarts) of water for making lemonade. They had only two jars, a three-quart and a five-quart jar, which were unmarked. How did they measure out exactly four quarts of water?
- (11) Two boys who weighed 100 pounds each and a man who weighed 200 pounds wished to cross a river. Their boat had a load limit of 200 pounds. How did they cross the river?
- (12) Mr. Brown had eight half dollars, one of which he knew was counterfeit and a little heavier than the others. Given a good set of balance scales, how could he determine which coin was counterfeit in only two weighings?
- (13) Joe had 5 pieces of chain of 3 links each. He wanted to join them to make a 15-link chain for his dog. The blacksmith charged 5¢ to cut a link and 10¢ to weld it together. How could he do the job for 45¢?

- (14) A farmer's wife took her eggs to market. She sold  $\frac{1}{3}$  of them at one store, and  $\frac{2}{3}$  of the remainder at another store. She then had 1 dozen eggs left. How many did she have at first?
- (15) Johnny had a sack of miniature toy cars. When lined up by twos there was 1 left over; when lined up by threes, there were 2 remaining; when lined up by fours, there were 3 left over; when lined up by fives, 4 remained; when lined up by sixes, there were 5 left over; but when lined up by sevens, they came out even. How many cars did Johnny have?
- (16) When the uncle of Bob, Bill, and Jerry Smith visited them, he brought in 21 coin containers, 7 of which were full of coins, 7 of which were half full, and 7 of which were empty. In order to keep the money and the containers, the boys were to divide the containers among themselves without shifting any coins from one container to another so that each would have the same number of coins and containers. How could they do this?
- (17) Buck needed to have his pony shod. The blacksmith said that he would charge 1¢ for setting the first nail, 2¢ for the second, 4¢ for the third, and would keep on doubling in this way until he had set all 32 nails. At this rate, how much would he charge for shoeing the pony?
- (18) Mary said to Sue: "If you'll give me 5 jacks, I'll have as many as you." Sue said: "If you'll give me 5 jacks, I'll have twice as many as you." How many jacks did each girl have?
- (19) In the Jones family, each daughter has the same number of brothers as sisters, and each son has twice as many sisters as brothers. How many children are in the family?
- (20) The Ray children's mother told them that they could each select one sale item from the toy counter. The toys sold for the same price (each cost more than 10¢) and the total bill was \$2.03. How many children were there, and how much did each toy cost?

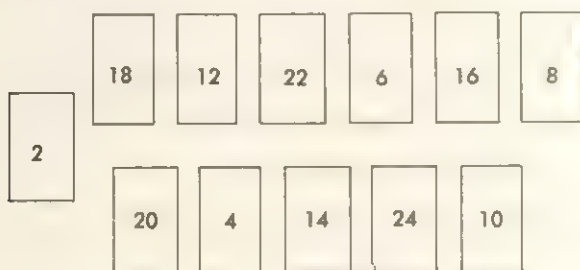
## PROBLEMS IN ARRANGEMENT AND CONSTRUCTION

Puzzles and games which involve manipulation of concrete and semiconcrete materials give opportunity for high-level thinking and for computation, yet require a minimum of pencil and paper work.

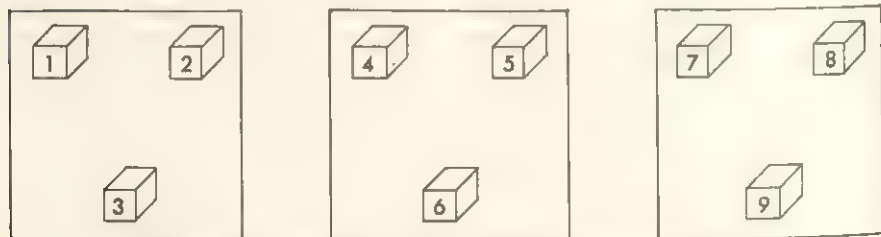
Model construction usually constitutes the problem; errors can be "erased" with a sweep of the hand.

In the following set of problems, numbered cards or blocks are used.

- (21) Arrange these cards in number pairs having the same total.



- (22) Move one block to make the totals of the numbers in each group equal.

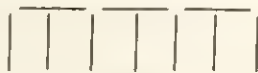


- (23) Stack blocks numbered 1 through 9 in groups of three so that their totals will be equal. (Magic squares may also be constructed in this way.)
- (24) Rearrange this square to form a figure with five blocks to a side.



- (25) A man planted ten trees. How did he plant them if he planted three rows with four trees in each row?

- (26) Farmer Brown made a sheep pen of 13 equal panels which looked like the figure at the right. One panel got broken. How could Farmer Brown rearrange the panels so that he could still have separate housing for each of his six sheep?



Toothpicks, matches, or other counters are used in a number of games in which the object is either to take, or not to take, the last counter. Children should be encouraged to analyze the rules and play to determine the principles on which the key to winning operates. For example, in the game which follows, the second person to draw may always win if he takes a number of counters which, combined with those of the first player, totals five at each drawing, since the total number of counters in play is 25.

Twenty-five counters are placed in a pile. Each player may, in turn, draw 1, 2, 3, or 4 counters. The player drawing the last counter wins.

## NUMBER PUZZLES

Number puzzles, creatively utilized, can be among the most worthwhile of mathematical recreations for upper-elementary grade children. Challenged to determine how and why a puzzle "works" or to devise puzzles of their own, children may gain insight into problem solving, and experience in constructing models (usually in the form of equations) of problem situations. The high interest value of such problems provides incentive for accurate computational procedures.

Some number puzzles are so constructed that the original number chosen is added to, subtracted from, etc., in such a way that the original number remains upon completion of computation. In others, the original number is eliminated through circuitous operations so that a predetermined answer remains.

The following puzzle is an example of this class of problem.

The teacher might present this puzzle as follows: Choose a number less than 10. Multiply it by 6. Add 9 to the product. Divide the sum by 3. Subtract 3. Tell me your answer, and I will tell you your number. Key: Divide the final number by 2.

Number chosen: 7

$$\begin{aligned} 6 \times 7 &= 42 \\ 42 + 9 &= 51 \\ 51 \div 3 &= 17 \\ 17 - 3 &= 14 \\ 14 \div 2 &= 7 \text{ (original number)} \end{aligned}$$

How and why it works:

$$\text{Original number} = N, \quad (N \times 6 + 9) \div 3 - 3 = \frac{6N + 9}{3} - 3 =$$

$$2N + 3 - 3 = 2N.$$

Divide by 2 to find N.

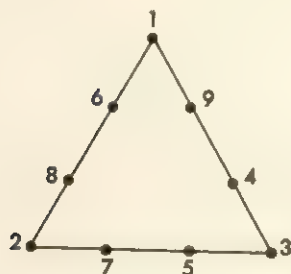
## SOLUTIONS TO NUMBERED PROBLEMS

- (1) Sum of each small triangle: 17;  
sum of each large triangle: 30;  
sum of each line of four numbers: 27.

(4)

8	1	6
3	5	7
4	9	2

- (2) (One solution)



(5)

$$\begin{array}{r} 234 \\ 567 \\ 495 \\ 201 \\ \hline 1497 \end{array}$$

(3)

$2\frac{1}{2}$	$\frac{3}{4}$	2
$1\frac{1}{4}$	$1\frac{3}{4}$	$2\frac{1}{4}$
$1\frac{1}{2}$	$2\frac{3}{4}$	1

(6)

$$\begin{array}{r} 746 \\ \times 352 \\ \hline 1492 \\ 3730 \\ 2238 \\ \hline 262592 \end{array}$$

(7)

$$\begin{array}{r} 935284 \\ -99999 \\ \hline 835285 \end{array}$$



- (15) 119. Number must be evenly divisible by 7, divisible by 2 with 1 remainder, divisible by 3 with 2 remainder, etc.
- (16) Three solutions are possible after determining how much coin each boy must have: 7 containers +  $3\frac{1}{2}$  containers =  $10\frac{1}{2}$ .

$$10\frac{1}{2} \div 3 (\text{boys}) = 3\frac{1}{2}.$$

One solution follows: Bob—3 empties, 1 half full, 3 full; Bill—1 empty, 5 half full, 1 full; Jerry—3 empties, 1 half full, 3 full.

- (17) \$42,949,672.95

- (18) (Most children would probably find this solution by trial and error.) The first statement establishes the condition that the numbers must have a difference of 10. The fact that Sue had at least 5 to give to Mary establishes the lower limits for the smaller number. Trying 5 and 15 as the two numbers, it is seen that these are too small, as the second transaction would give Mary 4 times as many jacks as Sue, rather than 2. The following is a more direct means of finding the two unknowns.

$X$  = Sue's jacks

$Y$  = Mary's jacks

1.  $X + 5 = Y - 5$

$X = Y - 10$

2.  $Y + 5 = 2(X - 5)$

$Y + 5 = 2(Y - 10) - 10$

$Y + 5 = 2Y - 20 - 10$

$Y = 35$

$Y - 10 = 25 = X$

Substituting  $Y - 10$  for  $X$

- (19) (Again, trial and error would be most children's approach.) A more direct method follows.

$X$  = number of girls

$Y$  = number of boys

$X - 1 = Y$

$2(Y - 1) = X$

$2Y - 2 - 1 = Y$

$Y = 3$

$X = 4$

$3 + 4 = 7$  children in the family

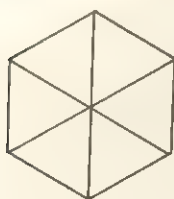
- (20) 7 children: 29¢ for each toy.
- (21) 2 and 24, 4 and 22, 6 and 20, etc.
- (22) Move the 9 to the first group.
- (23) One solution: 8, 1, 6; 3, 5, 7; 4, 9, 2.
- (24) A triangle, 5 to a side.

(25)

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      . . . .
      .
      .
      .
      . . . .
  
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(26) A hexagon



## EXERCISES

1. Work the problems given in this chapter.
2. What is the largest number that can be represented by two nines?
3. Using only the four fundamental processes, make (a) 1 with three 5's;  
(b) 1000 with eight 8's.
4. A farmer has sixteen horses. He has three stalls. He puts all of his horses in the three stalls with an odd number of horses in each stall. How does he do it?
5. Make Roman numerals with matchsticks. By moving one stick and replacing it somewhere in the array, make each of these statements true.  
(a)  $VII - I = VII$   
(b)  $IX + V = XXI$
6. Construct other exercises like those in problem 6.
7. To buy a watch priced at \$100 plus \$3 state tax, a man paid exactly 8 bills (paper money). This can be done with five twenties and three ones. Find another way.
8. You bought a number of items, each of equal price. You paid \$2.17. Each item cost more than 10¢. How many items did you buy and what was the unit price?
9. Think of a number. Double it. Add 4. Divide by 2. Subtract the number you thought of. Why is the answer 2?





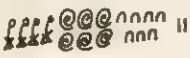


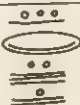
10. Discuss the role of mathematical recreations in mathematics education.
11. Develop some magic squares.

### Extended Activities

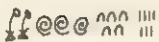

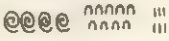



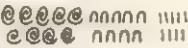

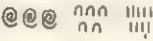

1. Construct a crossnumber puzzle, including the directions for working it.
2. Collect a bibliography of sources of mathematical recreational material including mathematical games, puzzles, riddles, historical mathematical procedures, etc.
3. Construct a pan balance. Show several uses to which it can be put in teaching arithmetic.

# Solutions for Selected Exercises

## Chapter 1

1.	Egyptian	Greek	Mayan	Roman
(a)	nn 	kg		XXVII
(b)	 II	umlv		CCCXLII
(c)	 II	d'xplv		IVDCLXXII
(d)	 II	n'g' & na		LVI CMVI

2.

	Egyptian	Mayan	Roman
(a)			MMMCCCLVII
(b)			CDXCVI
(c)			MCI
(d)			CMXCIX
(e)			CCCLIX

## 3. Positional

5. (a) 1215  
 (b) 943  
 (c) 2536  
 (d) 1646

11. (a)

12. (b)

13. (d)

14. (b)

15. (a)

16. (d)

## Chapter 2

1. (a) We are no more than 48 hours from anywhere in the world by commercial airline.  
 (b) The train stopped only seconds from the brink.  
 (c) It is a morning's journey from here.
2. (a) Measuring air pressure in a tire with an air gauge.  
 (b) Measuring water with a meter calibrated in gallons.  
 (c) Measuring intelligence with a test (and recording it with a score).
3. foot, ell, cubit, fathom, pace

4. (a) 180;  $1\frac{1}{4}$   
 (b)  $586\frac{2}{3}$ ;  $\frac{1}{3}$   
 (c)  $\frac{3}{875}$ ;  $\frac{1}{1000}$   
 (d) 29.6; 14.8  
 (e) 24,500; 56  
 (f) 11,400;  $2\frac{1}{4}$
8. Both. Pressure is derived measure (force per unit area) and the instrument measures it indirectly.
9. See pp. 42-43 for the answer to the first part.
10. The  $3\frac{1}{2}$  cups of flour implies accuracy to nearest  $\frac{1}{2}$  cup or a tolerance of  $\frac{1}{4}$  cup. Multiplication by 100 could generate an error equivalent to 25 cups of flour. The 1 cup of milk implies an error not exceeding  $\frac{1}{2}$  cup. Multiplication by 100 could lead to an error of 50 cups of milk. The relative error in the original flour measurement is  $\frac{\frac{1}{4}}{3\frac{1}{2}} = \frac{1}{14}$ . This relative error stays constant in the multiplication. The relative error in the milk measurement is  $\frac{1}{2}$ , and it also remains constant.
12. Can be answered in conjunction with Extended Activity 2 of this chapter.
13. (c)
14. (d)
15. (a) direct (but probably obtained indirectly)  
 (b) derived  
 (c) indirect  
 (d) direct  
 (e) derived
16. (b)
17. (a)

### Chapter 3

4. (a)  $13\frac{1}{8}$  cm.  
 (b) 30 in.  
 (c)  $15/91$  ft.  
 (d) 4.9 cm. (approx.)

### Chapter 4

1. (a)  $16 - 7$  is divisible by 3  
 (b)  $54 - 28$  is divisible by 13  
 (c)  $129 - 3$  is divisible by 9  
 (d)  $(-62 - 56)$  is divisible by 27

2. It is given that  $16 \equiv 4 \pmod{3}$ . This means that  $16 - 4 = 4 \cdot 3$ . Therefore,  $16 \cdot 5 - 4 \cdot 5 = 4 \cdot (3 \cdot 5)$ , and this implies that  $16 \cdot 5 - 4 \cdot 5$  is divisible by  $3 \cdot 5$ , or that  $16 \cdot 5 \equiv 4 \cdot 5 \pmod{3 \cdot 5}$ . Similarly,  $16 \cdot 5 - 4 \cdot 5 = 4 \cdot 5(3)$ . This implies that  $16 \cdot 5 - 4 \cdot 5$  is a multiple of 3, or that  $16 \cdot 5 \equiv 4 \cdot 5 \pmod{3}$ .

3. +	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Remember:  $7 \equiv 0 \pmod{7}$

4. (a) 2  
 (b) 12 is the direct sum, but  $12 \equiv 5 \pmod{7}$   
 (c) 6  
 (d) 16 is the direct difference, but  $16 \equiv 2 \pmod{7}$   
 (e) 3  
 (f) 33 is the direct sum, but  $33 \equiv 5 \pmod{7}$   
 (g) 121 is the direct sum, but  $121 \equiv 2 \pmod{7}$

5. $\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Remember:  $7 \equiv 0 \pmod{7}$

6. (a) 4  
 (b) 34 is the straightforward product, but  $34 \equiv 6 \pmod{7}$   
 (c) 206 is the straightforward product, but  $206 \equiv 3 \pmod{7}$
7. (a)  $6 \times 3 = 18$ , and  $18 \equiv 4 \pmod{7}$  since  $18 - 4$  is divisible by 7.  
 (b)  $16 \times 3 = 48$ , and  $48 \equiv 34 \pmod{7}$  since  $48 - 34$  is divisible by 7.  
 (c)  $24 \times 15 = 360$ , and  $360 \equiv 206 \pmod{7}$  since  $360 - 206$  is divisible by 7.

8. $\times$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Remember:  $8 \equiv 0 \pmod{8}$

9. (a) 0  
 (b) 4  
 (c) 64, and  $64 \equiv 0 \pmod{8}$   
 (d) 716, and  $716 \equiv 4 \pmod{8}$   
 (e) 240, and  $240 \equiv 0 \pmod{8}$
10. (a) 2 because  $3 \times 4 \equiv 6 \pmod{7}$   
 (b) 4 because  $5 \times 4 \equiv 6 \pmod{7}$
11. (a) 4  
 (b) 2  
 (c) 0  
 (d) 6  
 (e) 5

12.	+	0	1	2	3	4	5	6	7	8	9	10
	0	0	1	2	3	4	5	6	7	8	9	10
	1	1	2	3	4	5	6	7	8	9	10	0
	2	2	3	4	5	6	7	8	9	10	0	1
	3	3	4	5	6	7	8	9	10	0	1	2
	4	4	5	6	7	8	9	10	0	1	2	3
	5	5	6	7	8	9	10	0	1	2	3	4
	6	6	7	8	9	10	0	1	2	3	4	5
	7	7	8	9	10	0	1	2	3	4	5	6
	8	8	9	10	0	1	2	3	4	5	6	7
	9	9	10	0	1	2	3	4	5	6	7	8
	10	10	0	1	2	3	4	5	6	7	8	9

Addition Table  
 modulo 11

Multiplication Table  
modulo 11

$\times$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	1	3	5	7	9
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	6	10	3	7
5	0	5	10	4	9	3	8	2	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5
7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	3
9	0	9	7	5	3	1	10	8	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

The addition and subtraction exercises of #4, done modulo 11:

- (a) 9
- (b) 19
- (c) 10
- (d) 20
- (e) 6
- (f) 106
- (g) 458

The multiplication exercises of #6, done modulo 11:

- (a) 7
- (b) 37
- (c) 349

The division exercises of #11, done modulo 11:

- (a) 9
- (b) 2
- (c) 0
- (d) 9
- (e) 4

13. When  $a$  is divided by  $m$  with quotient  $q_1$  and remainder  $r_1$ , this division fact may be stated as

$$a = mq_1 + r_1$$

Similarly,

$$b = mq_2 + r_2$$

It is given that  $a - b$  is divisible by  $m$ . That is,

$$a - b = km \text{ for some integer } k.$$

Thus, substitution for  $a$  and  $b$ ,

$$a - b = (mq_1 + r_1) - (mq_2 + r_2) = km$$

$$(A) \quad m(q_1 - q_2) + (r_1 - r_2) = km.$$

It is given that  $0 \leq r_1 < m$ ,  $0 \leq r_2 < m$  (each is non-negative, making it greater than or equal to zero, and each is smaller than  $m$ ). This makes  $r_1 - r_2$  numerically smaller than either  $r_1$  or  $r_2$ , so, if it is positive or negative, it cannot be an integral multiple of  $m$ . This would make equation (A) false, since the other two terms are multiples of  $m$ . So, equation (A) can be true only if  $r_1 - r_2 = 0$  or  $r_1 = r_2$ .

14. (a)  $53 - 117 = -64$  and  $-64$  is divisible by 8.

$$\begin{array}{r} 53 = 6 \cdot 8 + 5 \\ 117 = 14 \cdot 8 + 5 \end{array} \quad \text{The remainders of 5 are equal.}$$

- (b)  $53 - (-117) = 170$  and 170 is divisible by 34.

$$\begin{array}{r} 53 = 1 \cdot 34 + 19 \\ -117 = -4 \cdot 34 + 19 \end{array} \quad \text{The non-negative remainders are equal}$$

15.  $a \equiv b \pmod{m}$  means  $a - b = K \cdot m$  for some integer  $K$ .

$$c \equiv d \pmod{m} \text{ means } c - d = L \cdot m \text{ for some integer } L.$$

$$\text{Subtract: } (a - c) - (b - d) = (K - L)m.$$

But this is just what  $a - c \equiv b - d \pmod{m}$  means: that  $(a - c) - (b - d) = \text{some multiple of } m$ .

16. Any integer can be expressed in terms of its positional meaning as follows:

$$I = d_n 10^n + d_{n-1} 10^{n-1} + \dots + d_2 10^2 + d_1 10^1 + d_0, \text{ in which}$$

$d_0 = \text{units digit}$

$d_1 = \text{tens digit}$

$d_2 = \text{hundreds digit, etc.}$

But this can be written as

$$\begin{aligned} I &= d_0 + d_1(9 + 1) + d_2(99 + 1) + d_3(999 + 1) + \dots \\ &= d_0 + d_1 + d_2 + d_3 + \dots + d_1 \cdot 9 + d_2 \cdot 99 + d_3 \cdot 999 + \dots \\ &= (d_0 + d_1 + d_2 + d_3 + \dots) + 9(d_1 + d_2 \cdot 11 + d_3 \cdot 111 + \dots) \\ I &= (d_0 + d_1 + d_2 + d_3 + \dots) = 9(d_1 + d_2 \cdot 11 + d_3 \cdot 111 + \dots) \end{aligned}$$

By definition then

$$I \equiv d_0 + d_1 + d_2 + d_3 + \dots \pmod{9} \text{ since their difference is a multiple of 9.}$$

$$\begin{array}{r} 17. \quad 01110_{\text{two}} = 14_{\text{ten}} \quad 10^{10}_{\text{two}} = 2^5_{\text{ten}} = 32_{\text{ten}} \\ \quad 10011_{\text{two}} = 19_{\text{ten}} \\ \hline \text{Add: } 100001_{\text{two}} = 33_{\text{ten}} \end{array}$$

Having converted to the decimal system, we are to show that

$$33 \equiv 1 \pmod{32}.$$

This is true since  $33 - 1$  is a multiple of 32.

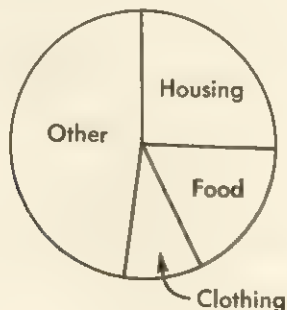
## Chapter 5

1. a. (a), (b), (e)  
b. (g), (i), (l)  
c. (o), (p), (r)  
d. (s), (v), (w)
2. (a), (b), (e), (h)
3. (c), (d)  
a. right angle  
b. acute angle  
c. acute angle
4. (b), (d)
5. a.  $\overline{AB}$   
b.  $\angle FEB$  or  $\angle BEF$   
d.  $\triangle ABD$   
e. E  
g. C

## Chapter 6

1.  $83\frac{1}{3}$
2. 27
3. (a) 98, 96, 95, 94, 91, 91, 87, 86, 86, 84, 84, 84, 84, 83, 83, 82, 81, 81, 79, 78, 78, 77, 77, 75, 74, 72, 72, 71, 65, 63, 52  
(b) 52 to 98  
(c) 82  
(d) 80.7

7.



11. (a) 1962  
 (b) From Jan., 1952, to Jan., 1962, about 425 points.  
 (c) It declined.
12. You know only that he was above the average. You do not know whether he was highest scorer or below the highest. You do not know even that more than half made less than he did. For example, the scores could have been:

501 scores of 83	$501 \times 83$	41,583
1 score of 82	$1 \times 82$	82
480 scores of 75	$480 \times 75$	36,000
15 scores of 20	$15 \times 20$	300
2 scores of 17	$2 \times 17$	34
1 score of 1	$1 \times 1$	1
<b>TOTAL</b> 1000	<b>AVG.</b> 78	78,000 total

15. No. The conclusions about the finite set may be overthrown by the very next observation.
16. Yes. A single observed case which does not conform to the behavior stated by the proposition proves that the proposition is invalid for this case, and hence proves it is not valid for all cases—that is, the proposition as stated is false.

## Chapter 7

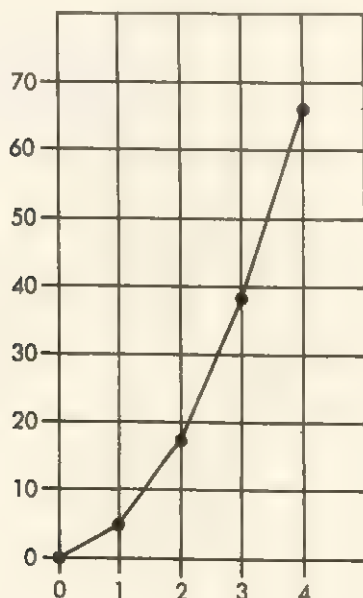
1.



Frequency

- |      |     |                 |   |
|------|-----|-----------------|---|
| 2. C | 256 | Major third:    | $\frac{5}{4}$ , Ratio of E to C = $\frac{324}{256} = \frac{81}{64}$   |
| D    | 288 |                 |   |
| E    | 324 | Perfect fourth: | $\frac{4}{3}$ , Ratio of F to C = $\frac{342}{256} = \frac{171}{128}$ |
| F    | 342 |                 |   |
| G    | 384 | Perfect fifth:  | $\frac{3}{2}$ , Ratio of G to C = $\frac{384}{256} = \frac{3}{2}$     |
| A    | 432 |                 |   |
| B    | 486 |                 |   |
| C    | 512 |                 |   |

3. About 6 million million miles, or about  $6 \times 10^{12}$  miles.
4. Between 16 and 17 seconds for the average diameter.
5. Due to atmospheric refraction of light. When the moon is near the horizon, its light passes through more air to reach the observer than when it is nearly overhead. This air serves as a magnifying glass.
6. Work = force  $\times$  distance. 600 ft. lbs. of work. One horsepower is 550 ft. lbs. per second. It is a rate at which work is done. For example, to move 55 lbs. through 10 ft. requires 550 ft. lbs. of work. The power necessary to do this work in 1 second is 1 horsepower.
7.  $\frac{1}{8}$  or 11.1% of water is hydrogen.  $\frac{16}{18}$  or 88.9% of water is oxygen.
16. 30 grams
17. 30% of the solution = 15 pts.  
10% of the solution = 5 pts.  
100% of the solution = 50 pts. (Answer)
18. (a)  
(1) 161.6°F  
(2) 122°F  
(3) 572°F
- (b)  
(1) 37.8°C  
(2) 100°  
(3) 22.2°
- 19.



20. (a) 49 cm.  
(b) 5 sec.
21.  $d = 4t^2$
22. (a) 18.1 cm.; 5.6 cm.; 24.8 cm.  
(b) Salt retards, fertilizer stimulates growth.  
(c) Some plants may not have been watered or given sunlight, etc.
23. 8 hrs. 20 min.
25. 2; 4
26. 12.5 millionths (approx.). The reason for the statement that the answer is approximate is that the information given from which the answer was obtained was only approximate.

## Chapter 8

2.  $9''$
3. (a)  $\left(\frac{5}{5}\right)$   
(b)  $888 + 88 + 8 + 8 + 8$
4. 

14	1	1
----	---	---

 When the protest is made that 14 is not an odd number, say, "Don't you think 14 is an odd number of horses to put in one stall?"
5. (a)  $\overbrace{VI + I} = VII$   
(b)  $\overbrace{XX + I} = XXI$
7. 1 \$1 bill, 1 \$2 bill, 5 \$10 bills, and 1 \$50 bill
8. 7 items for 31¢ each
9. Step 1:  $x$   
Step 2:  $2x$   
Step 3:  $2x + 4$   
Step 4:  $x + 2$   
Step 5:  $x + 2 - x = 2$

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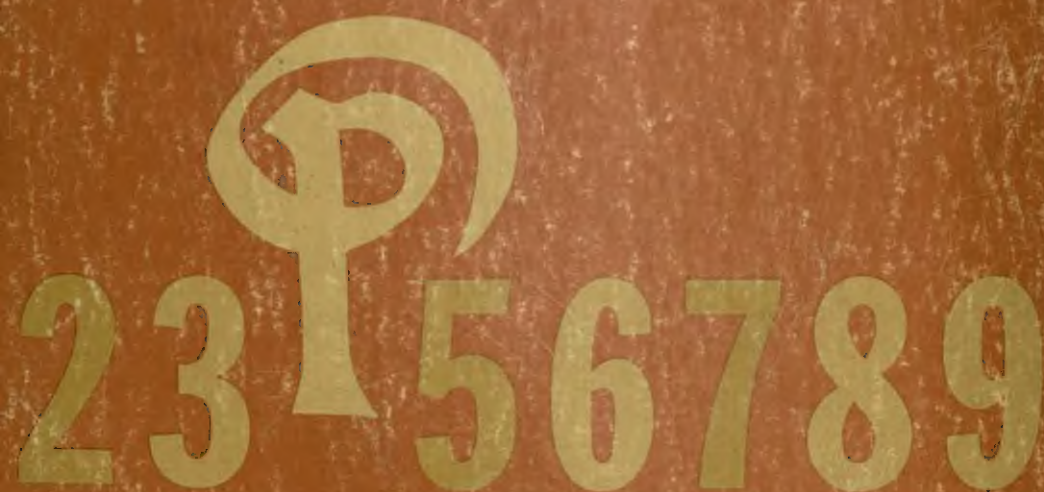
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